A categorification of Morelli's theorem

Goal: to show

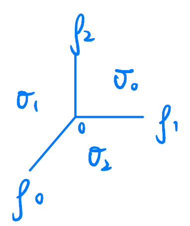
Categories of Θ - sheaves

Define a set $P(\Sigma, M)$ the element in $P(\Sigma, M)$ is a pair

$$(\sigma, x) \in P(\Sigma, M)$$

$\sigma \in \Sigma$ & x is a integral coset of $\sigma^\perp$

Eg. fan of $\mathbb{P}^2$



$P(\Sigma_{\mathbb{P}^2}, M) = \{ (\sigma_i, (a,b))$ $(\rho_0, (a_0, a_0) + \rho_0^\perp)$ [$\rho_0^\perp = \{x=-y\}$] $(\rho_1, (a_1, 0) + \rho_1^\perp)$ [$\rho_1^\perp = \{x=0\}$] $(\rho_2, (0, a_2) + \rho_2^\perp)$ [$\rho_2^\perp = \{y=0\}$] $(0, M_{\mathbb{R}}) \}$

a partial order by setting: $(\sigma, \phi) \leq (\tau, \psi)$ satisfies either of the following equ. conditions:

- $\tau \subset \sigma$ and if $\bar{\Phi}$ denotes the image of ϕ in $M_{\mathbb{R}}/\tau^\perp$ then $\bar{\Phi} - \psi \in \tau^\vee$
- The subsets $\phi + \sigma^\vee \subset M_{\mathbb{R}}$ and $\psi + \tau^\perp \subset M_{\mathbb{R}}$ have $\phi + \sigma^\vee \subset \psi + \tau^\vee$

Eg. in $\mathbb{P}^2$ $(\sigma_0, \phi) \leq (\sigma_0, \phi') \iff$ denote $\phi = (a,b)$ $\phi' = (a', b')$

$a \geq a'$ $\quad b \geq b'$

$(\rho_i, x_i) \leq (\rho_i, x_i') \iff$ denote $x_i = (a,0) + \rho_i^\perp$ or (a,a) or $(0,a)$ $\quad x_i' = (a',0) + \rho_i^\perp$ or (a',a') or $(0,a')$

$a \geq a'$

$(\rho_1, \Phi) \geq (\sigma_0, \psi) \iff \phi = (a,0) + \{x=0\}$ $\quad \psi = (b, c)$

$b \geq a$

(ρ_i, ϕ_i) . $(\sigma_i, \psi_i) \leq (0, M_{\mathbb{R}})$

$\rightsquigarrow$ a dg category $\Gamma(\Sigma, M)_R$

$$Ob(\Gamma(\Sigma, M)_R) = P(\Sigma, M)$$

$$Hom((\sigma,\phi),(\tau,\psi)) = \begin{cases} R & (\sigma,\phi) \subseteq (\tau,\psi) \\ 0 & \text{otherwise} \end{cases}$$

$$Hom((\tau,\psi),(\nu,\chi)) \otimes_R Hom((\sigma,\phi),(\tau,\psi)) \to Hom((\sigma,\phi),(\nu,\chi))$$

$$1 \otimes 1 \mapsto 1$$

Def: $\Theta(\sigma, \chi) = \hat{j}_{(\chi+\sigma^\vee)^\circ !}\, \omega_{(\chi+\sigma^\vee)^\circ}$

$\cap$

$Sh_c(M_{\mathbb{R}})$ the constandard sheaf on $M_{\mathbb{R}}$ associated to the open set $(\chi, \sigma^\vee)^\circ \subset M_{\mathbb{R}}$

where $\hat{j}_{(\chi+\sigma^\vee)^\circ} : (\chi+\sigma^\vee)^\circ \hookrightarrow M_{\mathbb{R}}$

Eg ($\mathbb{P}^2$) $(\chi_1+\sigma_0^\vee)^\circ$ like  $\hookrightarrow M_{\mathbb{R}}$

$(\chi_1+\rho_1^\vee)^\circ$ like  $\hookrightarrow M_{\mathbb{R}}$.

Def: $\Theta'(\sigma, \chi) = \hat{j}_* \mathcal{O}_\sigma(\chi)$ $\quad \hat{j} : X_\sigma \hookrightarrow X$

$\mathcal{O}_\sigma(\chi)$ the associated quasi-coherent sheaf to $R[(\chi+\sigma^\vee)\cap M]$

Eg. ($\mathbb{P}^2$) $\Theta'(\sigma_0, (0,0)) = \hat{j}_* \mathcal{O}_{U_0}$

$$\hat{j} : U_0 \to \mathbb{P}^2$$

$\Theta(\sigma, \chi) \overset{1:1}{\longleftrightarrow} \Theta'(\sigma, \chi)$ & $\langle\Theta\rangle \overset{\text{quasi equi.}}{\longleftrightarrow} \langle\Theta'\rangle$

Theorem 3.4. Let X be a toric variety with fan Σ, Let $\langle\Theta\rangle \subset Sh_c(M_{\mathbb{R}})$ denote the full triangulated subcategory generated by the objects $\Theta(\sigma, \chi)$. Let $\langle\Theta'\rangle \subset Q_T(X)$ denote the fully triangulated subcategory generated by the objects $\Theta'(\sigma, \chi)$. There exists a quasi-equivalence of dg category $K : \langle\Theta'\rangle \to \langle\Theta\rangle$ with the following properties:

- $k(\Theta'(\sigma, x)) \cong \Theta(\sigma, x)$
- If $(\sigma, \phi) \leq (\tau, \psi)$ in $P(\Sigma, M)$ then the map:

$$\operatorname{Ext}^0(\Theta'(\sigma,\phi), \Theta'(\tau,\psi)) \longrightarrow \operatorname{Ext}(\Theta(\sigma,\phi), \Theta'(\tau,\psi))$$

induced by k carries the canonical generator of the source to the canonical generator of the target.

Prop. 3.3. Let $(\sigma, \phi), (\tau, \psi) \in P(\Sigma, M)$

Then (1) $\operatorname{Ext}^i(\Theta(\sigma,\phi), \Theta(\tau,\psi)) \simeq \begin{cases} R & \text{if } i=0 \ \& \ (\sigma,\phi) \leq (\tau,\psi) \\ 0 & \end{cases}$

(2) $\operatorname{Ext}^i(\Theta'(\sigma,\phi), \Theta'(\tau,\psi)) \simeq \begin{cases} R & \text{if } i=0 \ \& \ (\sigma,\phi) \leq (\tau,\psi) \\ 0 & \end{cases}$

$$\omega_{(\phi+\sigma^\vee)^\circ} = or_{(\phi+\sigma^\vee)^\circ}[\dim(\phi+\sigma^\vee)^\circ] = or_{(\phi+\sigma^\vee)^\circ}[\dim M_{\mathbb{R}}]$$

$$R\mathcal{H}om(j_{(\phi+\sigma^\vee)^\circ !}\,\omega_{(\phi+\sigma^\vee)^\circ}, j_{(\psi+\tau^\vee)^\circ !}\,\omega_{(\psi+\tau^\vee)^\circ}) = \underline{R}_{(\phi+\sigma^\vee)^\circ}$$ orientational diff.

$$= R\mathcal{H}om(j_{(\phi+\sigma^\vee)^\circ !}\underline{R}_{(\phi+\sigma^\vee)^\circ}, j_{(\psi+\tau^\vee)^\circ !}\underline{R}_{(\psi+\tau^\vee)^\circ})$$

$$\cong R\mathcal{H}om(\underline{R}_{(\phi+\sigma^\vee)^\circ}, j^!_{(\phi+\sigma^\vee)^\circ} j_{(\psi+\tau^\vee)^\circ !}\underline{R}_{(\psi+\tau^\vee)^\circ})$$

$$\cong R\Gamma((\phi+\sigma^\vee)^\circ, j^*_{(\phi+\sigma^\vee)^\circ} j_{(\psi+\tau^\vee)^\circ !}\underline{R}_{(\psi+\tau^\vee)^\circ})$$

$$= R\Gamma((\phi+\sigma^\vee)^\circ, j_{(\psi+\tau^\vee)^\circ !}\underline{R}_{(\psi+\tau^\vee)^\circ})$$

check by stalk.

(2)

$$\begin{tikzcd} X_{\sigma\cap\tau} \arrow[r, hook] \arrow[d, hook] & X_\tau \arrow[d] \\ X_\sigma \arrow[r] & X \end{tikzcd}$$

$\rightsquigarrow$ dg functors $P(\Sigma, M)_R \hookrightarrow Sh(M_{\mathbb{R}})$ full dg embedding

$P(\Sigma, M)_R \hookrightarrow Q_T(X)$

$\Rightarrow \langle\Theta'\rangle \xleftarrow{\sim} \operatorname{Tr}(P(\Sigma, M)_R) \xrightarrow{\sim} \langle\Theta\rangle$

Coro 3.5. X, Σ, The dg functor κ defines a fully embedding of $\mathrm{Perf}_T(X)$ into $Sh_c(M_{\mathbb{R}})$

$\mathrm{Perf}_T(X) = \{ E \subset Q_T(X) \mid E$ is quasi-isomorphic to a bounded complex of T-equvriant vector bundle on each affine chart $\}$

T-variety X: a normal variety with a faithful T-action $\forall\, 1 \neq t \in T$ $tx \neq x$ $x \in X$

T-equvriant vector bundle

$$\begin{tikzcd} \sigma^* M \arrow[r, "\theta", "\sim"'] \arrow[d] & p_X^* M \arrow[r, shift left] \arrow[r, shift right] & M \arrow[d] \\ G \times X \arrow[rr, shift left, "p_X"] \arrow[rr, shift right, "\sigma"'] & & X \end{tikzcd}$$

satisfies

$m : \underline{G \times G} \times X \to G \times X$

$b : G \times \underline{G \times X} \to G \times X$

$p_{23} : G \times G \times X \to G \times X$

$m^*\theta = b^*\theta \circ pr_{23}^*\theta$

suffices to show $\mathrm{Perf}_T(X) \subset \langle \Theta' \rangle$

let $\mathcal{F}$ be a perfect complex on X

$\mathcal{F}$ quasi-isomorphic to (Čech resolution)

$$\bigoplus_{i_0} \hat{j}_{C_{i_0}*} \mathcal{F}|_{X_{C_{i_0}}} \longrightarrow \bigoplus_{i_0<i_1} \hat{j}_{C_{i_0 i_1}*} \mathcal{F}|_{X_{C_{i_0 i_1}}} \longrightarrow \cdots$$

$C_{i_l} = \mathrm{spec}(R[\sigma_{i_l}^\vee \cap M])$ $\quad C_{i_0 \cdots i_k} := C_{i_0} \cap \cdots \cap C_{i_k}$

reduce to show $j : U \hookrightarrow X$ U a T-stable affine chart.

$\forall$ T-equivariant vector bundle $\mathcal{E} \to U$

$j_* \mathcal{E} \in \langle \Theta' \rangle$

On any affine toric variety, a T-equivariant vector bundle splits as a sum of line bundle as $\mathcal{O}(x_\alpha)$

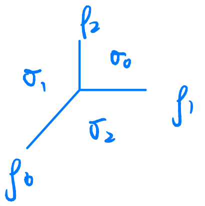
#

Def For each twisted polytope $\underline{x}$, define $P(\underline{x}) \in Sh_c(M_{\mathbb{R}})$ to be the cochain complex

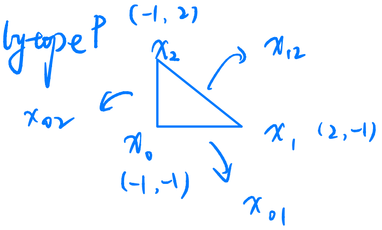
canonical map $a_0 - a_1$ ...

$$\bigoplus_{i_0} \Theta(C_{i_0}, x_{i_0}) \longrightarrow \bigoplus_{i_0<i_1} \Theta(C_{\sigma_{i_0 i_1}}, x_{i_0 i_1}) \to$$

Eg. $\mathbb{P}^2$ fan



polytope P



$$P(\underline{x}) = \left[\bigoplus_{i=0}^{2} j_{(x_i+\sigma_i^\vee)^\circ}\, \omega_{(x_i+\sigma_i^\vee)^\circ} \longrightarrow \bigoplus_{\substack{0\le s<t\le 2\\ l\neq s,t}} j_{(x_{st}+\rho_l^\vee)^\circ}\, \omega_{(x_{st}+\rho_l^\vee)^\circ} \right.$$

$$\left. \to \mathbb{R}_{M_{\mathbb{R}}}[2] \right]$$

Def Polytope: for a divisor $D = \sum a_i D_{\sigma_i}$, $\sigma_i \in \Sigma$

$P_D = \{ m \in M \mid \langle m, n_{\sigma_i} \rangle \geq -a_i \ \forall n_{\sigma_i} \text{ generator of } \sigma_i \}$

the convex hull of $\{x_1, \dots, x_v\}$

for $P = P_{-K_X}$ $\quad -K_X = \sum_{\rho_i \in \Sigma(1)} D_{\rho_i}$

$\mathcal{O}_X(\underline{x}) = \mathcal{O}_X(-K_X)$ is ample

Thm 3.7. $P(\underline{x})$ and P as above. If $\mathcal{O}_X(\underline{x})$ is ample

the $P(\underline{x}) \cong j_!(\omega_{P^\circ}) = j_!(\text{or}_{P^\circ}[\dim M_{\mathbb{R}}])$

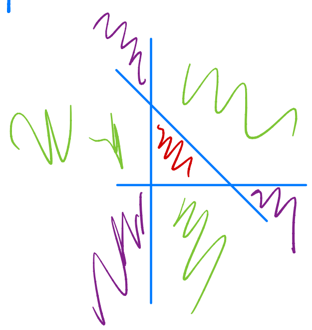
proof: $P^\circ \hookrightarrow x_i + (C_i^\vee)^\circ \rightsquigarrow j_!\, \omega_{P^\circ} \to \Theta(C_i, x_i)$

$\rightsquigarrow j_!\, \omega_{P^\circ} \to P(\underline{x})$

check on stalk

$$\left(\bigoplus_{i_0} \Theta(C_{i_0}, x_{i_0})\right)_x \to \left(\bigoplus_{i_0<i_1} \Theta(C_{i_0 i_1}, x_{i_0 i_1})\right)_x \to \cdots$$

In $\mathbb{P}^2$ case



$x \in P^\circ$

$\underline{R} \oplus \underline{R} \oplus \underline{R} \to \underline{R} \oplus \underline{R} \oplus \underline{R} \to \underline{R}$

0

$x \notin P^\circ$

$0 \to R \to R$

$1 \mapsto 1$

$x \notin P'$

$R \to R \oplus R \to R$

$1 \mapsto (1, 1)$

$(a, b) \mapsto (a - b)$

Def: we say a map $f: N_1 \to N_2$ is fan-preserving if for each $\sigma_1 \in \Sigma_1$ the image of σ_1 under $N_{1,\mathbb{R}} \to N_{2,\mathbb{R}}$ lies in another cone $\sigma_2 \in \Sigma_1$

f induces $\qquad T_i := N_{i,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}^*$

a map $f \otimes 1_{\mathbb{G}_m} : T_1 \to T_2$

a map $f^\vee : \sigma_2^\vee \cap M_2 \to \sigma_1^\vee \cap M_1$ with $f(\sigma_1) \subset \sigma_2$

a map $u_{f,\sigma_1,\sigma_2} : X_{\sigma_1} \to X_{\sigma_2}$

a map $u = u_f : X_1 \to X_2$ assembled from the u_{f,σ_1,σ_2}

equivariant w.r.t. $f \otimes 1_{\mathbb{G}_m} : T_1 \to T_2$

$v = v_f : M_{2,\mathbb{R}} \to M_{1,\mathbb{R}}$

Thm 3.8. (Functoriality) Let f be a fan-preserving map from $\Sigma_1 \subset N_{1,\mathbb{R}}$ to $\Sigma_2 \subset N_{2,\mathbb{R}}$. Suppose f satisfies

(1) $f^{-1}(\sigma_2) = \bigsqcup \sigma_i' \qquad \sigma_2 \in \Sigma_2 \qquad \sigma_i' \in \Sigma_1$

(2) f inj

Then (1) $u^* : Q_{T_2}(X_2) \to Q_{T_1}(X_1)$ takes $\langle \Theta' \rangle_2$ to $\langle \Theta' \rangle_1$

(2) $v_! : Sh_c(M_{2,\mathbb{R}}) \to Sh_c(M_{1,\mathbb{R}})$ takes $\langle \Theta \rangle_2$ to $\langle \Theta \rangle_1$

(3)
$$\begin{tikzcd} \langle \Theta' \rangle_2 \arrow[r, "K_2"] \arrow[d, "u^*"'] & \langle \Theta \rangle_2 \arrow[d] \\ \langle \Theta' \rangle_1 \arrow[r, "K_1"] & \langle \Theta \rangle_2 \end{tikzcd}$$
commutes up to natural iso.

proof: $\forall\ \sigma_2 \in \Sigma_2$

$$\begin{tikzcd} u^{-1}(X_{\sigma_2}) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & X_{\sigma_2} \arrow[d, hook, "\hat{j}_2"] \\ X_1 \arrow[r, "u"] & X_2 \end{tikzcd}$$

Fix $x_2 \in M_2$ $\quad x_1 = v(x_2)$

$$u^* \Theta'(\sigma_2, x_2) = u^* \hat{j}_* \mathcal{O}_{\sigma_2}(x_2)$$
$$= \hat{j}_{u^{-1}(X_{\sigma_2})*}\, u|^*_{u^{-1}(X_{\sigma_2})} \mathcal{O}_{\sigma_2}(x_2)$$

Assume $\sigma_{i_0} \cdots \sigma_{i_k}$ are the maximal cones in $f^{-1}(\sigma_2)$

$\Rightarrow$ $u^{-1}(X_{\sigma_2})$ = subspace generated by $\tau_1 \cdots \tau_k$

i.e. has affine char $B_1 \cdots B_k$

then $\hat{j}_{u^{-1}(X_{\sigma_2})*}\, u|^*_{u^{-1}(X_{\sigma_2})} \mathcal{O}_{\sigma_2}(x_2)$

$$= [\oplus \hat{j}_{B_{i_0}*} \underbrace{u|^*_{B_{i_0}} \mathcal{O}_{\sigma_2}(x_2)}_{\substack{\shortparallel \\ \mathcal{O}_{\sigma_{i_0}}(x_1)}} \to \bigoplus_{i_0 < i_1} \hat{j}_{B_{i_0 i_1}*}\, u|^*_{B_{i_0 i_1}} \mathcal{O}_{\sigma_{i_0 i_1}}(x_2) \to \cdots]$$

By $x_1 = v(x_2)$ & $u|_{B_{i_0}} = u_{f, \sigma_{i_0}, \sigma_2} : B_{i_0} \to X_{\sigma_2}$

this proves (1)

for (2) & (3) only need to show

$$\begin{tikzcd}
\langle \Theta' \rangle_2 \arrow[r, "k_2"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\circlearrowleft"] & sh_c(M_{2,\mathbb{R}}) \arrow[d, "v_!"] \\
\langle \Theta' \rangle_1 \arrow[r, "k_1"] & sh_c(M_{1,\mathbb{R}})
\end{tikzcd}$$

only need to show the natural quasi-iso.

$$l : \quad v_! \circ k_2 \xrightarrow{\sim} k_1 \circ u^*$$

suffice to give maps

$$l_{(\sigma_2, x_2)} \quad v_!(k_2(\Theta'(\sigma_2, x_2))) \to k_1(u^*(\Theta'(\sigma_2, x_2)))$$

with 1. $\forall (\sigma_2, x_2)$ $\quad l_{(\sigma_2, x_2)}$ is a quasi-iso

2° for $(\sigma_2, x_2) \le (\tau_2, y_2)$ in $P(\Sigma, M)$

$$\begin{tikzcd}
v_!(k_2(\Theta'(\sigma_2, x_2))) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & v_!(k_2(\Theta'(\tau_2, y_2))) \arrow[d] \\
k_1(u^*(\Theta'(\sigma_2, x_2))) \arrow[r] & k_1(u^*(\Theta'(\tau_2, y_2)))
\end{tikzcd}$$

$v_!(k_2(\Theta'(\sigma_2, x_2)))$

$= v_!(\Theta(\sigma_2, x_2))$

$= v_! \, j_{(x_2+\sigma_2^\vee)^\circ !} \, \omega_{(x_2+\sigma_2^\vee)^\circ} \quad = j_{(v(x_1)+v(\sigma_2^\vee))^\circ !} \, (v|_{(x_2+\sigma_2^\vee)^\circ})_! \, \omega$

(by f-inj)

$$\begin{tikzcd}
(x_2+\sigma_2^\vee)^\circ \arrow[r, "\text{submersion}"] \arrow[d] \arrow[dr, bend right=60] & (v(x_1)+v(\sigma_2^\vee)) \arrow[d] \\
M_{2,\mathbb{R}} \arrow[r] & M_{1,\mathbb{R}}
\end{tikzcd}$$

$$v_!(k_2(\Theta'(\sigma_2, x_2))) \underset{\text{quasi-iso}}{\xrightarrow{\sim}} j_{(v(x_2)+v(\sigma_2^\vee))^\circ !} \, \omega_{(v(x_2)+v(\sigma_2^\vee))^\circ}$$

$\cap$
$M_{1,\mathbb{R}}$

$k_1\, u^* \,\Theta' (\, \sigma_2 , x_2)$

$$= k_1 ([\, \bigoplus_{i_0} \Theta' (B_{i_0} , x_1) \longrightarrow \bigoplus_{i_0 < i_1} \Theta' (B_{i_0 i_1} , x_1) \longrightarrow \cdots]$$

$$= [\, \bigoplus_{i_0} \Theta (B_{i_0} , x_1) \longrightarrow \bigoplus_{i_0 < i_1} \Theta (B_{i_0 i_1} , x_1) \longrightarrow \text{---}]$$

defined $\quad \iota_i : \quad v_i\, k_2 (\qquad) \longrightarrow k_1\, u^* (\qquad)$

by $\quad \hat{J}(v(x_2) + v(\sigma_2^\vee))^\circ ; \quad {}^{w}(\, v(x_2) + v(\sigma_2^\vee))^\circ \qquad$ Thm 3.7

$$\downarrow$$

$$\bigoplus_{i_0} \Theta (B_{i_0} , x_1) \longrightarrow \bigoplus_{i_0 < i_1} \Theta (B_{i_0 i_1} , x_1) \longrightarrow \text{---}$$

denote $x_1 = v(x_2)$

by $\quad \underline{(x_1 + v(\sigma_2^\vee))^\circ \subset (x_1 + \tau_{i_0}^\vee)^\circ}$

like Thm 3.7.

Eg. 3.10. $\qquad f_p : N \to N \quad$ multiplication by p

3.11. $\qquad \Sigma_1$ refine $\Sigma_2 \quad \rightsquigarrow \quad u : X_1 \to X_2$ resolution

$$k \circ u^* = k$$

$$\mathcal{F} \otimes \mathcal{G} \longrightarrow k(\mathcal{F}) \otimes k(\mathcal{G})$$

$$
\begin{tikzcd}
\langle \Theta' \rangle \arrow[r, "k"] \arrow[d, phantom, "\cong"] & \langle \Theta \rangle \arrow[d, phantom, "="] \\
Q_T^{fin}(X) \arrow[r, "\sim"] \arrow[d, phantom, "\cup"] & Shard(M_{\mathbb{R}}, \Lambda_\Sigma) \arrow[d, phantom, "\cup"] \\
Perf_T(X) \arrow[r, "\sim"] & Sh_{cc}(M_{\mathbb{R}}, \Lambda_\Sigma) \arrow[d, "D"] \\
 & Sh
\end{tikzcd}
$$

$M_{\mathbb{R}}$, $N_{\mathbb{R}}$

Def : $\Gamma(M_{\mathbb{R}})$ denote the set of pairs (σ, c) where σ is a polyhedral cone in $N_{\mathbb{R}}$ and c is a coset in $M_{\mathbb{R}}/\sigma^{\perp}$

(a fan Σ in $N_{\mathbb{R}}$ $\Gamma(\Sigma, M) \subset \Gamma(M_{\mathbb{R}})$)

partial order $(\sigma, c) \le (\tau, d)$ by $c + \sigma^{\vee} \subset d + \tau^{\vee}$

$$\Theta(\sigma, c) = j_! \, \omega_{(c+\sigma^{\vee})^{\circ}} \qquad j : (c+\sigma^{\vee})^{\circ} \hookrightarrow M_{\mathbb{R}}$$

$\pi : T^*M \to M$ $\quad \pi^* \mathcal{F}$

Define shard sheaves. singular support :

- A finite shard sheaf on $M_{\mathbb{R}}$ is a constructible sheaf F whose singular support belong to a finite shard arrangement.
- A finite shard sheaf has type Z if its singular support belongs to a finite shard arrangement of type Z.
- finite shard arrangement is a subset $\Lambda \subset M_{\mathbb{R}} \times N_{\mathbb{R}}$

$$\Lambda = \bigcup_{i=1}^{n} Z_i$$

$Z_1, \cdots Z_n$ is a finite list of shards

If $\forall i$, Z_i is a lagrangians shard of Z then we say Λ has type Z.

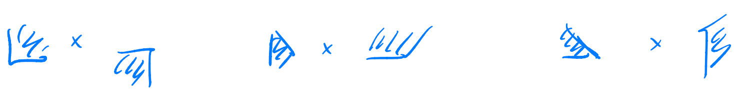
- For each $(\sigma, c) \in \Gamma(M_{\mathbb{R}})$ the lagrangian shard $Z(\sigma, c)$ is the subset of $M_{\mathbb{R}} \times N_{\mathbb{R}}$ given by

$$Z(\sigma, c) = c + \sigma^{\perp} \times -\sigma$$
$$= \{ (m, n) \mid -n \in \sigma \text{ and } \langle m - x, n' \rangle = 0 \text{ all } x \in c, n' \in \sigma \}$$

- The height of a shard $Z(\sigma, c)$ is the dimension of σ

shard$(M_{\mathbb{R}}, Z)$ = the full subcategory of finite ___ .

case $\mathbb{P}^1$ $\quad (\sigma_i, (0,0)) \quad Z(\sigma_i, (0,0)) = \sigma_i^{\vee} \times -\sigma_i$



a sheaf on $M_{\mathbb{R}}$ polyhedral if it is constructible w.r.t. a piecewise-linear stratification of $M_{\mathbb{R}}$. $\rightsquigarrow Sh_{c,pol}(M_{\mathbb{R}})$

Prop 4.1. The functor

$$V_x : Sh_{c,pol}(M_{\mathbb{R}}) \to Sh_{c,\mathbb{R}_{>0}}(T_x M_{\mathbb{R}})$$

is the unique function with the following property: for every $F \in Sh_{c,pol}(M_{\mathbb{R}})$ $\exists$ nbhd U of x in $M_{\mathbb{R}}$.

$$F|_U \cong V_x(F)|_{U'}$$

$$U' = \{ y \in T_x M_{\mathbb{R}} \mid y + x \in U \}.$$

$$T_x M \hookrightarrow TM \xrightarrow{\pi} M$$

define $\mathcal{FT}(F)$ on V^* with stalk $\mathcal{FT}(F)_\xi$ in dis-δ

$$\mathcal{FT}(F)_\xi \to \Gamma(V; F) \to \Gamma(\{v \in V \mid \xi(v) < -1\}, F|_{\{\xi(v)<-1\}})^{[+1]}$$

for a conical sheaf F on V.

Def. 4.3. Let $F \in Sh_{c,pol}(M)$, $x \in M$, and $\xi \in T^*_x M$

- The microlocalization of F at x is given by

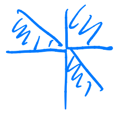
$$\mu_x(F) = \mathcal{FT}(V_x(F))$$

$$(Sh_{c,pol}(M) \to Sh_{c,\mathbb{R}_{>0}}(T^*_x M))$$

- The microlocal stalk of F at (x,ξ) is the stalk of $\mu_x F$ at ξ.

$$\mu_{x,\xi} F := (\mu_x(F))_\xi$$

- singular support of F $\quad SS(F) = \{ (x,\xi) \in T^*M \mid \mu_{x,\xi} F \neq 0 \}$

$\Lambda =$  $\times M_{\mathbb{R}}$

$\mathcal{F} \subset \mathrm{Shard}(M_{\mathbb{R}}, \Lambda)$

$j : N_{\mathbb{R}} \hookrightarrow M_{\mathbb{R}} \times M_{\mathbb{R}}$ $\quad i : M_{\mathbb{R}} \hookrightarrow N_{\mathbb{R}} \times M_{\mathbb{R}}$

$\mathrm{supp}\,(j^* i_* \mathcal{F} \otimes \quad (\mathcal{F}))$

If $\mathcal{Z} = \{ (\sigma, x) \mid Z(\sigma, x) \subset \Lambda \}$ $\quad \mathrm{Shard}(M_{\mathbb{R}}, \Lambda) := \mathrm{Shard}(M_{\mathbb{R}}, \mathcal{Z})$

Prop 5.1. For each $(\sigma, C) \in \Gamma(M_{\mathbb{R}})$ the constand and sheaf $\Theta(\sigma, C)$ is a finite shard sheaf.

proof:

Thm 5.2. Suppose $\mathcal{Z} \subset \Gamma(M_{\mathbb{R}})$ satisfies:

(Z1) The set of cones $\sigma \subset N_{\mathbb{R}}$ s.t. (σ, C) appears in $\mathcal{Z}$ for some C is a finite polyhedral fan

(Z2) If $(\sigma, x + \sigma^{\perp}) \in \mathcal{Z}$ and τ is a face of σ the $(\tau, x + \tau^{\perp}) \in \mathcal{Z}$

Then the category $\mathrm{Shard}(M_{\mathbb{R}}, \mathcal{Z})$ is generated by the sheaves $\{ \Theta(\sigma, C) \mid (\sigma, C) \in \mathcal{Z} \}$

for any a rational polyhedral fan Σ,

$$\Lambda_{\Sigma} = \bigcup_{\tau \in \Sigma} (\tau^{\perp} + M) \times -\tau$$ is a locally finite shard arrangement.

let $\mathcal{Z} = \{ (\sigma, x) \mid Z(\sigma, x) \subset \Lambda_{\Sigma} \}$ we have

$$Sh_c(M_{\mathbb{R}}, \Lambda_{\Sigma}) \supset \mathrm{Shard}(M_{\mathbb{R}}, \mathcal{Z}) \supset Sh_{cc}(M_{\mathbb{R}}, \Lambda_{\Sigma})$$

$$\mathrm{Shard}(M_{\mathbb{R}}, \Lambda_{\Sigma}) = \mathrm{Shard}(M_{\mathbb{R}}, \mathcal{Z})$$

full subcategory constructible objects which have compact support

$\mathcal{Z}$ satisfies (Z1) (Z2)

by Thm 5.2. There is a quasi-equivalence. $\mathrm{Shard}(M_{\mathbb{R}}, \Lambda_{\Sigma}) \simeq \langle \Theta \rangle$

Def 5.3. $\mathcal{F}$ a finite chard sheaf on $M_{\mathbb{R}}$ of height h, and let σ be an h-dimensional cone in $N_{\mathbb{R}}$. Then σ is narrow relative to $\mathcal{F}$ if for any point $x \in M_{\mathbb{R}}$ we either have $\{x\} x - \sigma \subset SS(\mathcal{F})$ or $\{x\} x - \sigma^{\circ} \cap SS(\mathcal{F}) = \emptyset$.

Lemma 5.4. F a finite chard sheaf on $M_{\mathbb{R}}$ of height h, σ an h-dim cone is narrow w.r.t. F. C a coset of $\sigma^{\perp}$. Then for $\forall$ $x \in C$ $\eta \in -\sigma^{\circ}$ the natural map

$$\mathcal{H}om(F, \Theta(\sigma, C))_x \longrightarrow Hom(\mu_{x,\eta}(F), \mu_{x,\eta}(\Theta(\sigma, C)))$$

is a quasi-iso.

Def: F finite chard sheaf of height h, σ a cone narrow w.r.t. to F. (σ, C) is said to be blocked w.r.t. F if

$$Hom(F, \Theta(\sigma, C)) \to \underline{\mathcal{H}om}(F, \Theta(\sigma, C))_x$$

fails to be a quasi-iso for some $x \in C$.

Lemma 5.7. Let $\mathfrak{Z} \subset \mathcal{P}(M_{\mathbb{R}})$ satisfy $(\mathfrak{Z}1)$ $(\mathfrak{Z}2)$, F a finite sheaf of type $\mathfrak{Z}$ and height h. Then $\exists$ $(\sigma, C) \in \mathfrak{Z}$ s.t. σ is h-dim. $Hom(F, \Theta(\sigma, C)) \neq 0$ and (σ, C) is not blocked for F.

proof of Thm 5.2.

Let $\langle \Theta \rangle_{\mathcal{Z}} \subset Sh_c(M_{\mathbb{R}})$ be the full ~~triangulated~~ category generated by

$\{ \Theta(\sigma, c) \mid (\sigma, C) \in \mathcal{Z} \}$

by ($\mathcal{Z}$2) $\exists$ $x \in M$ $(0, x) \in \mathcal{Z}$

$Z(0, x + M_{\mathbb{R}}) = M_{\mathbb{R}} \times \{0\}$

any sheaf $\mathcal{F}$ of height 0 Then $SS(\mathcal{F}) \subset M_{\mathbb{R}} \times \{0\}$

$\Rightarrow$ $\mathcal{F}$ is constant $\Rightarrow$ $\mathcal{F}$ of height 0 is generated by the sheave $\{ \Theta(0, x) \mid (0, x) \in \mathcal{Z} \}$

Induction on height

claim: If $\mathcal{F}$ is of type $\mathcal{Z}$ and height $\le h$, we can find another sheaf $\mathcal{F}'$ and a map $\mathcal{F}' \to \mathcal{F}$ with:

- $\mathcal{F}'$ has height $< h$
- the cone on $\mathcal{F}' \to \mathcal{F}$ is generated by sheaves of the form $\Theta(\sigma, C)$ where each σ is h-dimensional and each (σ, C) belong to $\mathcal{Z}$.

proof of claim: define $\mathcal{F}$ has h-complexity $\le n$

if $SS(\mathcal{F})$ is contained in a union of shards, at most n of which have height h.

Induction on the h-complexity of $\mathcal{F}$.

$\mathcal{F}$ has h-complexity ≤ 0 then $\mathcal{F}$ has height $< h$ ✓

for the $\mathcal{F}$ has h-complexity $\le n$.

by Lemma 5.7. $\exists$ σ of dim h and $c \in M_{\mathbb{R}} / \sigma^{\perp}$ s.t. (σ, C) is not blocked for $\mathcal{F}$ and s.t $hom(\mathcal{F}, \Theta(\sigma, c)) \ne 0$.

$\mathcal{F} \to hom(\mathcal{F}, \Theta(\sigma, C))^* \otimes \Theta(\sigma, C) \to \mathcal{F}'' \xrightarrow{+1}$

Lemm 5.4 $\Rightarrow$ iso on $M_{x,\eta}$ $x \in C$ $\eta \in -\sigma^{\circ}$ $\Rightarrow$ $Z(\sigma, C) \notin SS(\mathcal{F}'')$

Def A complex of quasicoherent sheaves $\mathcal{F}$ on an R-scheme X has finite fibers if for each R-valued point $x: \operatorname{Spec} R \to X$ of X the image $x^*\mathcal{F}$ of $\mathcal{F}$ under the pullback

$$Q(X) \longrightarrow Q(\operatorname{Spec} R)$$

is perfect

X with a group action, then we say that an equivariant quasi-coherent sheaf has finite fibers if the underlying non-equ. sheaf does.

Remark $\mathcal{F}$ finite fibers $\Leftrightarrow$ $\hom(\mathcal{F}, x_* R)$ perfect R-module

$$\Uparrow$$

$$\hom_R(x^*\mathcal{F}, R) \simeq \hom(\mathcal{F}, x_* R)$$

when R is alg. closed field

$\mathcal{F}$ finite fibers $\Leftrightarrow$ $\operatorname{Tor}_i(\mathcal{O}_X/m_x, \mathcal{F})$ finite-dimensional

$\operatorname{Tor}_i(\mathcal{O}_X/m_i, \mathcal{F}) = 0$ for all but finite many $i \in \mathbb{Z}$

In fact $\operatorname{Perf}_T(X_\Sigma) \subset Q_T^{\text{fin}}(X)$

Thm 6.3. for any toric variety X there is a quasi-equivalence $\langle\Theta'\rangle \simeq Q_T^{\text{fin}}(X)$

Def 6.5. X, Σ. For each integer h let $X^{(h)} \subset X$ denote the open toric subvariety of X obtained by removing all T-orbits of codimension greater than h.

In fact $\{T\text{-orbit}\} \leftrightarrow \{\sigma \subset \Sigma\}$

& $\dim \mathcal{O}_\sigma + \dim \sigma = \dim_{\mathbb{R}} M_{\mathbb{R}}$

$$X^{(h)} = \bigcup_{\dim \sigma \le h} \mathcal{O}_\sigma$$

$j: X^{(h)} \hookrightarrow X$

We say a quasicoherent sheaf $\mathcal{F} \in Q_T(X)$ has height $\le h$ if the following equivalent condition are satisfied:

(1) $\mathcal{F} \to j_* j^* \mathcal{F}$ is a quasi-iso.

(2) $i^* \mathcal{F} = 0$ whenever i is the inclusion of a T-orbit of codimension $> h$

denote $\Theta'(\sigma, x) := i_* \mathcal{O}_{O_\sigma}(x)$ $i: O_\sigma \hookrightarrow X$ for $(\sigma, x) \in P(\Sigma, M)$

$\mathcal{F}$ of height $< h$ $\Leftrightarrow$ $\hom(\mathcal{F}, \Theta'(\sigma, x)) = 0$ $\dim \sigma = h$

Def: $\mathcal{F} \in Q_T(X)$ has finite fibers and of height $\leq h$,

$SS_h(\mathcal{F}) := \{ (\sigma, x) \in P(X, M) \mid \dim(\sigma) = h \ \hom(\mathcal{F}, \Theta'(\sigma, x)) \neq 0 \}$

$(\sigma, x) \in SS_h(\mathcal{F})$ is said unblocked for $\mathcal{F}$ if the map

$$\mathrm{Hom}(\mathcal{F}, \Theta'(\sigma, x)) \to \mathrm{Hom}(\mathcal{F}, \Theta'(\sigma, x))$$

is a quasi-isomorphism.

Lemma 6.7. Let $\mathcal{F} \in Q_T(X)$ have finite fibers and height $\leq h$. If $SS_h(\mathcal{F}) \neq \phi$ then $\exists$ $(\sigma, x) \in SS_h(\mathcal{F})$ is unblocked for $\mathcal{F}$.

proof of Thm 6.3.

a quasi-coherent sheaf $\mathcal{F}$ of height ≤ 0.

$\Rightarrow \mathcal{F} \xrightarrow{\sim} j_* j^* \mathcal{F}$ for $j: O_0 \hookrightarrow X$

$O_0 = \mathrm{Spec}(k[x_i^{\pm 1}])$

$\Rightarrow \mathcal{F}$ is of the form $\Theta'(0, x)$

Induction on h.

claim: If $\mathcal{F}$ has finite fibers and is of height $\leq h$ we can find another quasi coherent sheaf $\mathcal{F}'$ and a map $\mathcal{F}' \to \mathcal{F}$ with the following properties:

- $\mathcal{F}'$ has height $< h$
- the cone on $\mathcal{F}' \to \mathcal{F}$ is generated by sheaves of the form $\Theta'(\sigma, x)$ where each σ is h-dimensional.

proof of claim: Induction on the size of $SS_h(\mathcal{F})$.

If $SS_h(\mathcal{F}) = \phi$ $\Rightarrow$ $\mathcal{F}$ has height $< h$

suppose $SS_h(\mathcal{F})$ has n element and that we have proven for all $\mathcal{F}$ s.t. $SS_h(\mathcal{F}') < n$ (*)

by lemma 6.7, $\exists\ \sigma$ s.t. $\dim \sigma = h$ and $(\sigma, \chi) \in P(\Sigma, M)$ with $\mathrm{Hom}(\mathcal{F}, \Theta'(\sigma, \chi)) \neq 0$ and (σ, χ) is not blocked for $\mathcal{F}$.

$$\mathcal{F} \to \mathrm{Hom}(\mathcal{F}, \Theta'(\sigma,\chi))^* \otimes \Theta'(\sigma, \chi) \to \mathcal{F}'' \to$$

Apply $\mathrm{Hom}(-, \theta'(\tau, \xi))$

$$\mathrm{Hom}(\mathcal{F}'', \theta'(\tau, \xi)) \to \mathrm{Hom}(\mathcal{F}, \Theta'(\sigma, \chi)) \otimes \mathrm{Hom}(\Theta'(\sigma,\chi), \theta'(\tau, \xi))$$

$$\to \mathrm{Hom}(\mathcal{F}, \theta'(\tau, \xi)) \xrightarrow{+1}$$

when $(\tau, \xi) = (\sigma, \chi)$

$$\mathrm{Hom}(\mathcal{F}, \theta'(\tau, \xi)) \xleftarrow{\simeq} \mathrm{Hom}(\mathcal{F}, \Theta'(\sigma, \chi))$$

$\Rightarrow \mathrm{Hom}(\mathcal{F}'', \theta'(\sigma, \chi)) = 0 \Rightarrow (\sigma, \chi) \notin SS_h(\mathcal{F}'')$

$\Rightarrow \mathcal{F}''$ satisfies (*) #

Thm 6.8. X a toric variety corresponding to a fan Σ. Then there is a quasi-equivalence of dg-categories

$$\kappa : Q_T^{fin}(X) \xrightarrow{\sim} \mathrm{Shard}(M_{\mathbb{R}}, \Lambda_\Sigma)$$

the image of $\mathrm{Perf}_T(X_\Sigma) \subset Q_T^{fin}(X)$ under κ ?

Thm 7.1. Let X be a proper toric variety corresponding to a fan $\Sigma \subset M_{\mathbb{R}}$.

Let $\kappa : \langle \Theta' \rangle \to \langle \Theta \rangle$ be the functor.

(1) If $\mathcal{E} \in \langle \Theta' \rangle$ is perfect, then $\kappa(\mathcal{E}) \in \langle \Theta \rangle$ has compact support.

(2) The resulting functor $\mathrm{Perf}_T(X) \to Sh_{cc}(M_{\mathbb{R}}, \Lambda_\Sigma)$ is a quasi-equivalence.

[Seidel] every equivariant vector bundle on a smooth projective toric variety has a bounded resolution by line bundles.

show $\kappa(\mathcal{L})$ has compact support when $\mathcal{L}$ is a line bundle.

by $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$

$\mathcal{L}_1$ & $\mathcal{L}_2$ ample then $\kappa(\mathcal{L}_1)$ has compact support

by Thm 3.7. ($\underset{\text{Line bundle}}{\mathcal{O}_X(\chi)}$ ample $\underset{\kappa(\mathcal{O}_X(\chi))}{P(\chi)} \cong j_!(\omega_{p\circ})$)

by Thm 7.3. $\kappa(\mathcal{L}_2^{-1}) = -D(\kappa(\mathcal{L}_2))$

Thm 7.3. Suppose X is a complete toric variety and $\mathcal{E} \in \mathrm{Perf}_T(X)$

then $\exists$ natural quasi-isomorphism

$$\kappa(\underset{\mathcal{H}om(\mathcal{E}, \mathcal{O})}{\mathcal{E}^\vee}) \simeq -D(\kappa(\mathcal{E}))$$

Lemma 7.4. Let $F \in Sh_c(M_{\mathbb{R}})$ Suppose that F is polyhedral and has compact support. Then F is strongly dualizable w.r.t. the convolution product.

a sheaf on $M_{\mathbb{R}}$ polyhedral if it is constructible w.r.t. a piecewise-linear stratification of $M_{\mathbb{R}}$.

strongly dualizable $\qquad F \to F \circledast (-DF) \circledast F \to F \qquad$ identity.

proof of Thm 7.1. $\qquad$ for $\quad F \in Sh_{cc}(M, \Lambda_\Sigma)$

$\exists \ \mathcal{G} \in \langle \Theta' \rangle$ s.t. $\kappa(\mathcal{G}) \cong F$

suffice to show $\mathcal{G}$ is perfect.

$\Updownarrow$

strongly dualizable

by Lemma 7.4. $\quad \kappa(\mathcal{G}) = F$ strongly dualizable

$\exists \ \mathcal{H} \qquad \kappa(\mathcal{H}) = -DF$

$$\begin{array}{ccc} \kappa(\mathcal{G}) & & F \\ \downarrow & & \downarrow \\ \kappa(\mathcal{G} \otimes \mathcal{H} \otimes \mathcal{G}) & = & F \circledast (-DF) \circledast F \\ \downarrow & & \downarrow \\ \mathcal{G} & & F \end{array}$$

$\Rightarrow \ \mathcal{G}$ strongly dualizable