T-duality and homological mirror symmetry for toric varieties.

FLTZ.

I. HMS

Thm 1.2 X_Σ be a complete toric variety defined by fan $\Sigma \subset N_{\mathbb{R}}$. Then there is a quasi-equivalence:

$$\tau: \mathrm{Perf}_{\underset{\sim}{T}}(X_\Sigma) \xrightarrow{\sim} \mathrm{Fuk}(\underset{\substack{\shortparallel \\ N_{\mathbb{R}} \times M_{\mathbb{R}}}}{T^* M_{\mathbb{R}}}; \Lambda_\Sigma)$$

which intertwines tensor product on left hand side and $\underset{\sim}{\diamond}$ on the right hand side.

In same setting, we have

$$(*) \quad \bar{\tau}: \mathrm{Perf}(X_\Sigma) \longrightarrow \mathrm{Fuk}(T^* T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma)$$

is a quasi-embedding

As corollary, we have

$$D\mathrm{Coh}_T(X_\Sigma) \simeq D\mathrm{Fuk}(T^*M_{\mathbb{R}}; \Lambda_\Sigma)$$

$$(*)\ D\mathrm{Coh}(X_\Sigma) \hookrightarrow D\mathrm{Fuk}(T^*T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma)$$

Rmk: The $(*)$ is a equivalence proved by Kuwagaki.

This part is essentially the combination of NZ and FLTZ.

Thm 3.1.3.2

$$K.\ \mathrm{Perf}_T(X_\Sigma) \to Sh_{cc}(M_{\mathbb{R}}; \Lambda_\Sigma)$$

is a quasi-equivalence. sending equivariant ample line bundle $\mathcal{L}_{\vec{c}}$ on X_Σ to costandard constructible sheaf $i_! \omega_{\underset{\sim}{\Delta_{\vec{c}}^{\circ}}}$ on $M_{\mathbb{R}}$,

$$\mathcal{L}_{\vec{c}} = \mathcal{O}(D_{\vec{c}})$$

$$D_{\vec{c}} = \sum_{i=1}^{r} c_i D_i$$

D_i corresponds to v_i generator of rays

Then $\Delta_{\vec{c}} := \{ m \in M_{\mathbb{R}} \mid \langle m, v_i \rangle \geq -c_i, \ i = 1, \ldots, r \}$

when $\mathcal{L}_{\vec{c}}$ is ample, $\Delta_{\vec{c}}$ is a convex polytope

$$\bar{K} : \mathrm{Perf}(X_\Sigma) \to Sh_c(T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma)$$

is a quasi embedding

We define Fukaya category

$\mathrm{Fuk}(T^*X, \Lambda)$ where Λ is a conical

Lagrangian.

Ob: generated by Lagrangian L with

$L^\infty \subset \Lambda^\infty$ and $\overline{\pi(L)}$ is compact

$\pi: T^*X \to X$

$$L^\infty = \overline{\iota(L)} \cap S^*X$$

Here $\iota: T^*X \to D^*X$

$$(x, \xi) \to \left(x, \frac{\xi}{\sqrt{1+\|\xi\|^2}}\right)$$

$$S^*X = \{(x,\xi) \mid \|\xi\| = 1\}$$

Mor: Same

For embedding $i: Y \to X$ we have

$i_* \mathbb{C}_Y$ standard obj in $Sh_c(X)$

and $L_{Y,*} \subset T^*X$

$$L_{Y,*} = T_Y^*X + \Gamma_{df}$$

$f = \log m$ m nonnegative defining

function for ∂Y.

For costandard obj $i_! \omega_Y$ in $Sh_c(X)$,

we have costandard Lagrangian

$$L_{Y,!} := T^*_Y X - \Gamma_{df}$$

Thm 5.4 There is a quasi-equivalence

of A_∞-cats

$$\mu : Sh_{cc}(M_{\mathbb{R}}; \Lambda_\Sigma) \xrightarrow{\sim} Fuk(T^* M_{\mathbb{R}}; \Lambda_\Sigma)$$

and $\bar{\mu} : Sh_c(T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma) \xrightarrow{\sim} Fuk(T^* T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma)$

Then we investigate the monoidal structure.

$$\begin{aligned} \gamma_L : Fuk(T^*X) &\to mod(Fuk(T^*X))^{\circ} \\ P &\mapsto hom(P, -) \end{aligned}$$

Given L of $Fuk(T^*X_0 \times T^*X_1)$

we can define

$$\Psi_L : Fuk(T^*X_0) \to mod(Fuk(T^*X_1))^\circ$$

$$P \to hom_{Fuk(T^*X_0 \times T^*X_1)}(L, \alpha_{X_0}(P) \times -)$$

$$T^*X_0 \to T^*X_0$$

$$\alpha_{X_0} : (x, \xi) \to (x, -\xi)$$

$$\Phi_{K!} : Sh_c(X_0) \to Sh_c(X_1)$$

$$\mathcal{F} \to p_{1!}(K \otimes p_0^* \mathcal{F})$$

$$\begin{tikzcd} & X_0 \times X_1 \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ X_0 & & X_1 \end{tikzcd}$$

Thm: [Nadler]

Let $K \in Sh_c(X_0 \times X_1)$ and its microlocalization $L = \mu_{X_0 \times X_1}(K)$. Then there is a quasi-equivalence

$$\gamma \circ \mu_{X_1} \circ \Phi_{K!} \simeq \tilde{\Psi}_{L!} \circ \mu_{X_0}$$

Thm: 3.7. Given $X_1 = X_{\Sigma_1}$, $X_2 = X_{\Sigma_2}$ and fan preserving map $f : N_1 \to N_2$, where f

is $i_2 j$. $\mu: X_1 \to X_2$ $\nu: M_{2;\mathbb{R}} \to M_{1;\mathbb{R}}$

$L_\nu \simeq T^*_{T_2}(M_{2;\mathbb{R}} \times M_{1;\mathbb{R}})$ The following

diagram commutes.

$$\begin{tikzcd}
\mathrm{Perf}_{T_2}(X_2) \arrow[r, "K"] \arrow[d, "\mu^*"] & Sh_{cc}(M_{2;\mathbb{R}}; \Lambda_{\Sigma_2}) \arrow[r, "\simeq"] \arrow[d, "\nu_!"] & \mathrm{Fuk}(T^*M_{2;\mathbb{R}}; \Lambda_{\Sigma_2}) \arrow[d, "\Phi_{L_\nu !}"] \\
\mathrm{Perf}_{T_1}(X_1) \arrow[r, "K"] & Sh_{cc}(M_{1;\mathbb{R}}; \Lambda_{\Sigma_1}) \arrow[r, "\simeq"] & \mathrm{Fuk}(T^*M_{1;\mathbb{R}}; \Lambda_{\Sigma_1})
\end{tikzcd}$$

Let G be a Lie gp $(M_\mathbb{R})$

$$\nu(g_1 \cdot g_2) = g_1 \cdot g_2$$

Then we define

$$L_1 \diamond L_2 := \Phi_{L_\nu !}(L_1 \times L_2) \quad \text{in } \mathrm{Fuk}(T^*G)$$

$$\Downarrow$$

$$\underline{T_{\Gamma_\nu}(G \times G)}$$
$$\cap$$
$$\mathrm{Fuk}(T^*(G \times G) \times \underline{T^*G})$$

Prop 3.9. The microlocal functor

μ_G satisfies

$$\mu_G(- * -) = \mu_G(-) \diamond \mu_G(-)$$

$$F_1 * F_2 = v_! (F_1 \boxtimes F_2)$$

Pf: $\mu_G(F_1 * F_2) = \mu_G \circ v_! (F_1 \boxtimes F_2)$

$$\overset{(*)}{\simeq} \Phi_{L_v !} \circ \mu_{G\times G}(F_1 \boxtimes F_2)$$

$$= \Phi_{L_v !} \circ (\mu_G(F_1) \times \mu_G(F_2))$$

$$= \mu_G(F_1) \diamond \mu_G(F_2)$$

(*) is given by applying thm 3.5, to $K = C_{T_v}$

and $\Phi_{K!} \simeq v_!$

$G = \mathcal{M}_{\mathbb{R}}$ $\qquad \otimes \to \diamond$

II. T-duality

Thm 3. (Equivariant homological mirror symmetry is T-duality)

Let X_Σ be a non singular proj toric variety. Any equivariant line bundle $\mathcal{L}_{\vec{c}}$ with an <u>admissible</u> hermitian metric h the T-dual Lagrangian $\mathbb{L}_{\vec{c},h}$ is an object in $Fuk(T^*M_{\mathbb{R}}; \Lambda_\Sigma)$ and

$$\mathbb{L}_{\vec{c},h} \simeq \tau(\mathcal{L}_{\vec{c}})$$

where τ is in thm 1.

$$\mathcal{L}_{\vec{c}} = \mathcal{O}(D_{\vec{c}})$$

choose $s_i \in H^0(X, \mathcal{O}_X(D_i))$ vanishing on D_i exactly.

then $s^{\vec{c}} := \prod_{i=1}^{r} s_i^{c_i}$

section of $\mathcal{L}_{\vec{c}}$. Restriction of $\mathcal{L}_{\vec{c}}$ to

$$\underline{X_{\{0\}}} = X \setminus \bigcup_{i=1}^{n} \{D_i\} \simeq (\mathbb{C}^*)^n \quad \text{is}$$

a holomorphic framing.

Let $\mathcal{L}_{\vec{c}}$ has a <u>admissible</u> hermitian metric h, [real analytic / $T_{\mathbb{R}}$-invariant / defines a unitary connection with curvature non degenerate closed 2-form]

Let $\nabla_{\vec{c},h}$ be the unique connection on $\mathcal{L}_{\vec{c}}$ determined by h. Connection 1-form with respect to unitary frame $s^{\vec{c}}/\|s^{\vec{c}}\|_h$ of $\mathcal{L}_{\vec{c}}|_{X_{\{0\}}}$ is

$$\alpha = -2\sqrt{-1}\,\mathrm{Im}\left(\bar{\partial} \log \|s^{\vec{c}}\|_h\right)$$

For $X_{\Sigma,\mathbb{C}} = (\mathbb{C}^*)^n$ we gives coordinates $r_j e^{i\theta_j}$, then we have $\|s_{\mathbb{C}}\|_h$ is independent of θ_i by $T_{\mathbb{R}}$-invariance.

$$\sqrt{-1}\,\alpha = 2\,\mathrm{Im}(\bar\partial \log \|s_{\mathbb{C}}\|_h)$$

$$= 2\,\mathrm{Im}\Big(\sum_{j=1}^{n} \frac{\partial}{\partial r_j} (\log \|s_{\mathbb{C}}\|_h)(dr_j - \sqrt{-1}\, r_j d\theta_j)\Big)$$

$$= -\sum_{j=1}^{n} \Big(\frac{\partial}{\partial y_j} \log \|s_{\mathbb{C}}\|_h\Big)\, d\theta_j$$

$y_j = \log r_j$

$\underline{f_{\mathbb{C},h}(y) = -\log \|s_{\mathbb{C}}\|_h}$ is real analytic function in y

$$\sqrt{-1}\,\alpha = \sum_{j=1}^{n} \frac{\partial f_{\mathbb{C},h}(y)}{\partial y_j}\, d\theta_j$$

$\mathbb{L}_{\mathbb{C},h} \subset M_{\mathbb{R}} \times N_{\mathbb{R}}$ is the graph of the map $N_{\mathbb{R}} \to M_{\mathbb{R}}$ defined by

$$y \longrightarrow \sqrt{-1}\,\alpha\,|_{p_2^{-1}(y)}$$

$$\underbrace{p_2 : T_{\mathbb{R}} \times N_{\mathbb{R}}}_{\substack{\wr\downarrow \\ X_{(\sigma)} = (\mathbb{C}^*)^n \\ \shortparallel \\ U(1)^n \times \mathbb{R}^n}} \to N_{\mathbb{R}}$$

$$H^1(T_{\mathbb{R}}; \mathbb{R}) \simeq M_{\mathbb{R}}$$

$$\boxed{T^{\vee}_{\mathbb{R}} \times N_{\mathbb{R}} \to N_{\mathbb{R}}}$$

In term of coordinates γ_j on $M_{\mathbb{R}}$ and y_i on $N_{\mathbb{R}}$, $\mathbb{L}_{\vec{c},h}$ given by

$$\frac{\gamma_j}{2\pi} = \frac{\partial f_{\vec{c},h}}{\partial y_i}(y)$$

If $D_{\vec{c}'} - D_{\vec{c}}$ is principal divisor

then $\quad s_{\vec{c}'} = s_{\vec{c}} \prod_{i=1}^{n} t_j^{m_j}$

$$f_{\vec{c}',h} = f_{\vec{c},h} - \sum_{j=1}^{m} m_j y_j$$

$$\frac{\partial f_{\vec{c}',h}(y)}{\partial y_j} = \frac{\partial f_{\vec{c},h}(y)}{\partial y_j} - m_j$$

Therefore $\mathbb{L}_{\vec{c}',h} = \mathbb{L}_{\vec{c},h}$ in $\mathrm{Fuk}(T^* M_{\mathbb{R}})$

It remains to show $\mathbb{L}_{\tilde{c},h}$ is an object in $\mathrm{Fuk}(T^*M_{\mathbb{R}}, \Lambda_\Sigma)$ and

$\mathbb{L}_{\tilde{c},h} \simeq \tau(\mathcal{L}_{\tilde{c}})$

① $\mathcal{L}$ is tame $[N\mathbb{Z}]$

proved by estimation of differential of $f_{\tilde{c},h}$

② $\tau_c(L)$ is bounded. Later

③ $\mathcal{L}^\infty \subset \Lambda_\Sigma^\infty$ Section A.2

Pf of ②:

As $T_{\mathbb{R}}$ acts on X symplectic form

$\omega_h := \sqrt{-1} F_h$

on $\underline{X_{\{0\}} \simeq (\mathbb{C}^*)^n}$

$$\omega_h = \sqrt{-1}\, d\alpha = \sum_{j=1}^n d\left(\frac{\partial f_{\tilde{c},h}}{\partial y_j}\right) \wedge d\theta_j$$

then we have analytic moment map

$$\Phi_{\tilde{c},h}(y,\theta) = \sum_{j=1}^n \frac{\partial f_{\tilde{c},h}}{\partial y_j}(y)\, e^*_j$$

Let $x_j = \frac{\gamma_j}{2\pi}$ on $M_{\mathbb{R}}$

$$\mathbb{L}_{\vec{c},h} = \{(x,y) \in M_{\mathbb{R}} \times N_{\mathbb{R}} \mid x = \Phi_{\vec{c},h} \circ j_0(y)\}$$

$j_0 : N_{\mathbb{R}} \to X_\Sigma$

exp S^1

$$N \otimes \mathbb{R}^+ \hookrightarrow (\mathbb{C}^*)^n \simeq N \otimes \mathbb{C}^* = X_\Sigma - \cup D_i$$

$$\Delta_{\vec{c}} = \Phi_{\vec{c},h}(X_\Sigma)$$

$$\Psi_{\vec{c},h} = \Phi_{\vec{c},h} \circ j_0$$

$N_{\mathbb{R}} \to \Delta^\circ_{\vec{c}}$ is a diffeomorphism

$$\mathbb{L}_{\vec{c},h} = \{(x, \Psi^{-1}_{\vec{c},h}(x)) \mid x \in \Delta^\circ_{\vec{c}}\} \subset \Delta^\circ_{\vec{c}} \times N_{\mathbb{R}}$$

For $\mathcal{L}^{-1}_{\vec{c}} = \mathcal{L}_{-\vec{c}}$ we consider

$$\Delta_{-\vec{c}} = -\Delta_{\vec{c}}$$

$$\mathbb{L}_{-\vec{c},h^{-1}} = \{(\underline{-\Psi_{\vec{c},h}(y), y}) \mid y \in N_{\mathbb{R}}\}$$

$$= \alpha(\mathbb{L}_{\vec{c},h})$$

$\mathbb{L}_{\tau, h} = \tau(\mathcal{L}_\tau)$

We show $\mathbb{L}_{-\tau, h^{-1}} \simeq \mu(i_* \mathbb{C}_{\Delta^\circ_{-\tau}})$

By showing are the same in the

Yoneda embedding

$$Y : DFuk(T^* M_{\mathbb{R}}) \to \mathrm{mod}(DFuk(T^* M_{\mathbb{R}}))$$
$$\mathcal{L} \to \hom_{DFuk(T^* M_{\mathbb{R}})}(-, \mathcal{L})$$

We fix a triangulation of $M_{\mathbb{R}}$ containing

$\{ U_{\tau, -\tau} \mid \tau \in \Sigma \}$

$U_{\tau, \tau} = \{ m \in \Delta_\tau \mid \langle m, v_i \rangle = -c_i \iff v_i \in \tau \}$

E.g. $U_{\{0\}, \pm\tau} = \Delta^\circ_{\pm\tau}$

Yoneda modules of any obj $Y(L)$

is expressed in terms (sums and cones and shifts of) Yoneda modules from standards

$\gamma(\mu(i_* \mathcal{C}_T))$. So we only need to consider values of

$$\hom(\mu(i_* \mathcal{C}_T), \mathcal{L}) = \hom_{DFuk(T^*M_{\mathbb{R}})}(\underbrace{L_{\{t\}\times}}_{\text{fiber } T^*_t M_{\mathbb{R}}}, \mathcal{L})$$

as $T \in \mathcal{T}$ contractible.

Let $\mathcal{L} = \mathbb{L}_{-\vec{\tau}, h^{-1}}$, consider $T \neq \Delta^{\circ}_{-\vec{\tau}}$

Let $t \in T$. Then $T \cap \Delta^{\circ}_{-\vec{\tau}} = \emptyset$, clearly

$$\hom_{DFuk(T^*M_{\mathbb{R}})}(L_{\{t\}\times}, \mathbb{L}_{-\vec{\tau}, h^{-1}}) = 0$$

Otherwise, if $T \cap \partial\Delta$ is nonempty then we

$$\hom_{DFuk(T^*M_{\mathbb{R}})}(L_{\{t\}\times}, \mathbb{L}_{-\vec{\tau}, h^{-1}}) = 0$$

By prop 8.9. Basically it says by some flow we can separate those Lagrangians

Finally if $T = \Delta^{\circ}_{-\vec{\tau}}$, since $\mathbb{L}_{-\vec{\tau}, h^{-1}}$ is a graph over T, we have

$$\hom_{DFuk(T^*M_{an})}(L_{z\in S_x}, \mathbb{L}_{\vec{\tau},h^{-1}}) = \mathbb{C}$$

Therefore $\mathbb{L}_{\vec{\tau},h^{-1}} \simeq \mu(i_* \Delta^{\circ}_{-\vec{\tau}})$

The statement $\mathbb{L}_{\vec{\tau},h} \simeq \mathcal{C}(\mathcal{L}_{\vec{\tau}})$ follows from the compatibility of microlocalization and Verdier duality.