Global dimension of geometric stability condition
(based on arXiv:2408.00519 and work in progress in joint with Dongjian Wu)

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Contents

Outline

Background

Bridgeland stability was introduced by Bridgeland, inspired by Douglas's Π-stability. Let $\mathcal{D}$ be a triangulated category and $K(\mathcal{D})$ its Grothendieck group. Then a *Bridgeland pre-stability* is a pair $\sigma = (Z, \mathcal{P})$, with $Z : K(\mathcal{D}) \to \mathbb{C}$ called *central charge* and a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ called *slicing*, satisfying

1. if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi)$.

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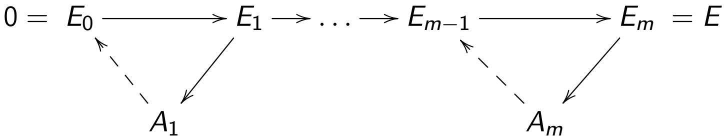
2. for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
3. if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$, then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$.

Background

4. for $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_m$ and a collection of triangles



with $A_i \in \mathcal{P}(\phi_i)$ for all $1 \leq i \leq m$, called *Harder-Narasimhan filtrations*. We denote $\phi^+(E) = \phi_1$ and $\phi^-(E) = \phi_m$.

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Fact: The datum $(Z, \mathcal{P})$ is equivalent to datum $(Z, \mathcal{A})$, where Z is central charge and $\mathcal{A}$ the heart of a bounded t-structure. From left to right, we take $\mathcal{A} = \mathcal{P}([0, 1))$. And from right to left, we take $\mathcal{P}(\phi)$, $\phi \in [0, 1)$ to be the set of semistable object $E \in \mathcal{A}$ with respect to Z and has $Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi)$. And extend to $\phi \in \mathbb{R}$ by property (2) in the definition of Bridgeland stability. So in latter part, I will use two notations interchangeably.

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From geometric perspective, we are particularly interested in case $\mathcal{D} = D^b(\mathrm{Coh}\ X)$. In this case, we may fix an ample line bundle H and choose $\Lambda \cong \mathbb{Z}^{\dim X+1}$ and $\nu(E) = (H^{\dim X-i}\,\mathrm{ch}_i(E))$.

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The existence of Bridgeland stability condition is not obvious from definition. A result of Kawatani shows that if X is relative affine with dimension greater than 1, then $\mathcal{D}$ admits no stability condition.

Background

On the other hand, for projective varieties, the existence of stability condition is established on curves (Bridgeland, Macrì, Okada), K3 surface (Bridgeland), general smooth surface (Arcara-Bertram), variety with full exceptional collections (Macrì), and many 3-folds including abelian threefolds (Bayer-Macrì-Stellari), Fano variety of Picard rank 1 and quintic threefolds (Li) and many other cases.

Stability space

After knowing its existence, a natural question is to determine the set of all possible stability conditions. The interesting facts is that the space of stability condition admits a natural (generalized) metric structure given by

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in D^b(X)} \left\{ \left|\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)\right|, \left|\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E)\right|, \|Z_1 - Z_2\| \right\}$$

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With above topology, the stability space also admit a structure of complex manifold follows from **Bridgeland deformation theorem**.

Theorem (Bridgeland 2007)

$$\mathcal{Z} : \mathrm{Stab}(X) \to \mathrm{Hom}(K(X), \mathbb{C})$$
$$(Z, \mathcal{P}) \to Z$$

is a local homeomorphism.

Stability space

Stab(X) admits a natural right $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. Let $(T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ where $f : \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(\phi + 1) = f(\phi) + 1$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ orientation preserving linear isomorphism such that induces map on $S^1 = \mathbb{R}/2\mathbb{Z} = (\mathbb{R}^2 \backslash \{0\})/\mathbb{R}_{>0}$ are the same. Then given $(Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$, we define $(Z', \mathcal{P}')$ by $Z' = T^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

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Stab(X) also admits a natural left Aut $D^b(X)$-action, given by $\phi(Z, \mathcal{P}) = (Z \circ \phi^{-1}, \mathcal{P})$.

Stability space

The structure of $\mathrm{Stab}(X)$ has much to do with properties of $D^b(\mathrm{Coh}\, X)$. For example, Bayer and Bridgeland determines the derived equivalence group of K3 surface of Picard rank 1 by proving contractibility of $\mathrm{Stab}(X)$. However, it is in general difficult to determine the global topology of stability space.

Global dimension function

Global dimension function is introduced by Qiu a generalization of homological dimension of abelian category. Let $\mathcal{P}$ be a slicing on a triangulated category $\mathcal{D}$. Then its *global dimension* is defined as

$$\operatorname{gldim}(\mathcal{P}) := \sup\{\phi' - \phi \mid \operatorname{Hom}(\mathcal{P}(\phi), \mathcal{P}(\phi')) \neq 0\}$$

For a stability condition $\sigma = (Z, \mathcal{P})$, the global dimension $\operatorname{gldim} \sigma$ is defined to be $\operatorname{gldim} \mathcal{P}$.

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Example: For an algebra A, $\mathcal{P}_A$ be the canonical slicing on $D^b(A)$, where $\mathcal{P}_A(0) = \mod A$ and $\mathcal{P}_A(0,1) = \emptyset$. Then we have $\operatorname{gldim}\mathcal{P}_A = \operatorname{gldim} A$.

Global dimension function

The gldim function has following properties:

1. gldim is continuous function on $\operatorname{Stab}\mathcal{D}$.
2. gldim is invariant under the $\mathbb{C}$-action (rotation and scaling) and the action of $\operatorname{Aut}\mathcal{D}$.

Therefore, we can define a function

$$\operatorname{gldim} : \operatorname{Aut}(\mathcal{D})\backslash \operatorname{Stab}\mathcal{D}/\mathbb{C} \to [0, +\infty]$$

The global dimension of $\mathcal{D}$ is given by

$$\operatorname{Gd}\mathcal{D} := \inf \operatorname{gldim} \operatorname{Stab}\mathcal{D}$$

Global dimension function

Qiu suggest an approach to prove the contractibility of stability space via global dimension function by following strategy:

1. If the subspace $\mathrm{gldim}^{-1}(\mathrm{Gd}\,\mathcal{D})$ is non-empty, then it is contractible. Moreover, the preimage $\mathrm{gldim}^{-1}([\mathrm{Gd}\,\mathcal{D}, x))$ contracts to $\mathrm{gldim}^{-1}(\mathrm{Gd}\,\mathcal{D})$ for any real number $x > \mathrm{Gd}\,\mathcal{D}$.
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Fan, Li, Liu and Qiu successively apply the strategy to $\mathbb{P}^2$-case. In particular, they show $\mathrm{gldim}^{-1}(2)$ is a large subset of geometric stability condition on $\mathbb{P}^2$.

Global dimension function

Our main result is the following.

Theorem (Wu and Z.)

Let X be a Fano threefold of Picard rank 1. Then global dimension of geometric stability condition $\sigma \in \Sigma_\Psi(\widetilde{GL}_2^+(\mathbb{R}) \times \Pi)$ is 3, where $\Pi = \{(\alpha, \beta, a, b) \in \mathbb{R}^4 \mid \alpha > 0, a > \frac{\alpha^2}{6} + \frac{\alpha}{2}|b|\}$.

Remark

It is obvious that the above subspace of $\mathrm{Stab}(X)$ is contractible. However, we don't show that above space is exactly $\mathrm{gldim}^{-1}(3)$.

Outline

Geometric stability condition

Definition

A stability condition σ on $D^b(X)$ is called *geometric* if for each point $p \in X$, the skyscraper sheaf $\mathcal{O}_p$ is σ-stable, and all skyscraper sheaves are of the same phase.

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A stability condition σ on $D^b(X)$ is called *geometric* if for each point $p \in X$, the skyscraper sheaf $\mathcal{O}_p$ is σ-stable, and all skyscraper sheaves are of the same phase.

In fact, the latter part, all skyscraper sheaves are of the same phase is redundant at least for numerical stability condition shown by Fu, Li and Zhao.

Geometric stability on surfaces

The set of geometric stability condition on surfaces is well studied by Bridgeland, Li and Dell.

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Theorem (Dell arXiv:2307.00815)

Let X be a smooth projective surface. Then

$$\mathrm{Stab}^{\mathrm{Geo}}(X) \cong$$
$$\mathbb{C} \times \{(H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbb{R}}(X) \times \mathrm{NS}_{\mathbb{R}}(X) \times \mathbb{R}^2 \mid \alpha > \Phi_{X,H,B}(\beta)\}$$

where $\Phi_{X,H,B}(x)$ is the Le Portier function defined as

$$\Phi_{X,H,B}(x) =$$
$$\limsup_{\mu \to x} \left\{ \frac{\mathrm{ch}_2(F) - B\,\mathrm{ch}_1(F)}{H^2\,\mathrm{ch}_0(F)} \mid F \text{ is } H\text{-semistable}, \mu_H(F) = \mu \right\}$$

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1. *F is taken over coherent sheaves on X not in the heart determined by Bridgeland stability.*

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2. *The $\mathbb{C}$ factor on the right hand side is from rotation and scaling such that $Z(\mathcal{O}_p) = -1$ for every $p \in X$.*

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1. *F is taken over coherent sheaves on X not in the heart determined by Bridgeland stability.*
2. *The $\mathbb{C}$ factor on the right hand side is from rotation and scaling such that $Z(\mathcal{O}_p) = -1$ for every $p \in X$.*
3. *By Bogomolov inequality $\mathrm{ch}_1^2(F) - 2\,\mathrm{ch}_0(F)\,\mathrm{ch}_2(F) \geq 0$, we have*

$$\Phi_{X,H,B}(x) \leq \frac{1}{2}[(x - \frac{HB}{H^2})^2 - \frac{B^2}{H^2}]$$

the stability condition with α greater than right hand side with x replace by β is already known in Bridgeland's paper on K3 surface.

Geometric stability on polarized threefolds

We generalize the result on surfaces to threefold. From now on, let X be a Fano threefold of Picard rank 1 and H ample generator of Picard group. As previously mentioned, it is in general difficult to construct stability condition with full group $K(X)$. Instead, we choose an ample divisor and consider a sublattice Λ_H generated by $(H^3 \operatorname{ch}_0(E), H^2 \operatorname{ch}_1(E), H \operatorname{ch}_2(E), \operatorname{ch}_3(E))$. We define $\operatorname{Stab}_H(X)$ to be the stability condition space where $Z : K(X) \to \mathbb{C}$ factor through Λ_H.

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Bayer, Macrì and Toda suggests a method to construct geometric stability condition on threefold via double tilting.

Recap: Tilting

Let $\mathcal{A}$ be an abelian category. The pair of additive subcategory $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if

1. $\mathrm{Hom}(T, F) = 0$ if $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
2. $\mathrm{Hom}(\mathcal{T}, Y) = 0$, then $Y \in \mathcal{F}$.
3. $\mathrm{Hom}(X, \mathcal{F}) = 0$, then $X \in \mathcal{T}$.
4. For every $A \in \mathcal{A}$, there exists an exact sequence

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$$0 \to T \to A \to F \to 0$$

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then $\langle \mathcal{F}[1], \mathcal{T} \rangle \subset D^b(\mathcal{A})$ is heart of a bounded t-structure on $D^b(\mathcal{A})$.

Recap: Tilting

Now let $\mathcal{A} = \mathrm{Coh}(X)$. And define slope function

$$\mu_\beta(E) = \begin{cases} +\infty & \mathrm{ch}_0^\beta(E) = 0 \\ \frac{H^2 \mathrm{ch}_1^\beta(E)}{H^3 \mathrm{ch}_0^\beta(E)} & \text{otherwise} \end{cases}$$

where $\mathrm{ch}^\beta(E) := e^{\beta H} \mathrm{ch}(E)$.

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where $\mathrm{ch}^\beta(E) := e^{\beta H}\,\mathrm{ch}(E)$. We define torsion pair

$$\mathcal{T}_\beta = \{E \in \mathrm{Coh}(X) \mid \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \mu_\beta(G) > 0\}$$

$$\mathcal{F}_\beta = \{E \in \mathrm{Coh}(X) \mid \text{any subsheaf } F \hookrightarrow E \text{ satisfies } \mu_\beta(F) \leq 0\}$$

We can show that $(\mathcal{T}_\beta, \mathcal{F}_\beta)$ form a torsion pair. And let $\mathrm{Coh}^\beta(X) := \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$.

Recap: Tilting

Now we can define a new stability function on $\mathrm{Coh}^{\beta}(X)$

$$\nu_{\alpha,\beta}(E) = \begin{cases} +\infty & \alpha^2 H^2 \mathrm{ch}_1^{\beta}(E) = 0 \\ \frac{\alpha H \mathrm{ch}_2^{\beta}(E) - \frac{1}{2}(\alpha H)^3 \mathrm{ch}_0^{\beta}(E)}{\alpha^2 H^2 \mathrm{ch}_1^{\beta}(E)} & \text{otherwise} \end{cases}$$

And define the torsion pair as before by replacing μ_β by $\nu_{\alpha,\beta}$. Then we obtain a new tilted heart, denoted by $\mathcal{A}_{\alpha,\beta}$.

Geometric stability on polarized threefolds

The $(\mathcal{A}_{\alpha,\beta}, Z_{\alpha,\beta} = -\mathrm{ch}_3^\beta + \alpha^2 H^2 \mathrm{ch}_1^\beta + i(\alpha H \mathrm{ch}_2^\beta - \frac{1}{2}(\alpha H)^3 \mathrm{ch}_0^\beta)$ is a stability condition if and only if the generalized Bogomolov inequality holds, i.e., For any ν_H-semistable object $E \in \mathrm{Coh}^\beta(X)$, satisfying $\nu_H(E) = 0$, we have

$$\mathrm{ch}_3^\beta(E) \leq \frac{(\alpha H)^2}{6} \mathrm{ch}_1^\beta(E)$$

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$$\operatorname{ch}_3^\beta(E) \leq \frac{(\alpha H)^2}{6} \operatorname{ch}_1^\beta(E)$$

Or equivalently, for every $E \in \operatorname{Coh}^\beta(X)$ $\nu_{\alpha,\beta}$-semistable,

$$\alpha^2 \overline{\Delta}_H(E) + 4(H \operatorname{ch}_2^\beta(E))^2 - 6H^2 \operatorname{ch}_1^\beta(E) \operatorname{ch}_3^\beta(E) \geq 0$$

Geometric stability on polarized threefolds

Now we define an analog of Le Portier function for threefold. We define

$$\Psi_{X,\nu}(\alpha,\beta,b) := \limsup_{\mu\to\nu}\left\{\frac{\mathrm{ch}_3^{\beta} - bH\,\mathrm{ch}_2^{\beta}}{H^2\,\mathrm{ch}_1^{\beta}(F)} \mid F \text{ is } \nu_{\alpha,\beta}\text{-semistable }, \nu_{\alpha,\beta}(F)=\mu\right\}$$

and

$$\Psi_X := \Psi_{X,0}$$

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And we denote by

$$\mathcal{B}_{\Psi} := \left\{ (\alpha, \beta, a, b) \in \mathbb{R}^4 \mid \alpha > 0, a > \max\{\frac{\alpha^2}{6}, \Psi_X(\alpha, \beta, b)\} \right\}$$

Geometric stability on polarized threefolds

Similar to Dell's result, we proved following theorem

Theorem (Wu and Z.)

Let (X, H) be a polarized threefold satisfying generalized Bogomolov inequality. Then there is a continuous open embedding

$$\Sigma_\Psi : \widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \times \mathcal{B}_\Psi \to \mathrm{Stab}_H^{\mathrm{Geo}}(X)$$
$$(g, (\alpha, \beta, a, b)) \to (Z_{\alpha,\beta}^{a,b}, \mathcal{A}_{\alpha,\beta})[g]$$

where $Z_{\alpha,\beta}^{a,b} = -\mathrm{ch}_3^\beta + bH\,\mathrm{ch}_2^\beta + aH^2\,\mathrm{ch}_1^\beta + i(H\,\mathrm{ch}_2^\beta - \frac{\alpha^2}{2}H^3\,\mathrm{ch}_0^\beta)$.

Geometric stability on polarized threefolds

Remark

1. *We can not prove that our space is full geometric stability space which is different from Dell's result.*
2. *If generalized Bogomolov inequality holds, then we have* $\Psi_X(\alpha, \beta, b) \leq \frac{\alpha^2}{6} + \frac{\alpha}{2}|b|$. *If a greater than right hand side, then the corresponding stability condition is constructed by Bayer, Macrì and Stellari.*
3. *If* (X, H) *is a polarized abelian threefold. We can show that*

$$\Psi_X(\alpha, \beta, b) = \frac{\alpha^2}{6} + \frac{\alpha}{2}|b|$$

which recover the result of Fu, Li and Zhao.

Outline

Sketch of proof

Now we give a sketch of proof of our main theorem.

Sketch of proof

The part $\mathrm{gldim}(\sigma) \geq 3$ is easy by considering $\mathcal{O}_x$ and

$$\mathrm{Ext}^3(\mathcal{O}_x, \mathcal{O}_x) = \mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x)^\vee \neq 0$$

For another direction, we claim that if E, F σ-semistable and $\phi(E) < \phi(F)$, then we have

$$\mathrm{Hom}(F \otimes L, E) = 0$$

for any $L = \mathcal{O}(cH), c > 0$, H ample generator of Picard group. If the claim is true, and suppose $\phi(F) > \phi(E) + 3$, by Serre duality, we have

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(F \otimes K_X^{-1}, E[3]) = 0$$

So we have $\mathrm{gldim}(\sigma) \leq 3$.

Sketch of proof

To prove the claim, we consider a family of stability conditions

$$(Z_t, \mathcal{P}_t) := \sigma^{a,b}_{\alpha,\beta-tc}, \quad t \in [0,1]$$

By generalized Bogomolov inequality, we can show that

$$\operatorname{Im}(Z_t'(F)\overline{Z_t(F)}) \geq 0$$

Sketch of proof

We then apply following theorem of Mozgovoy

Theorem (Mozgovoy arXiv:2201.08797)

Let $\sigma_t = (Z_t, \mathcal{P}_t)_{t\in[0,1]}$ be a continuous family of stability conditions on $\mathcal{D}$ such that map $Z : [0,1] \to \Lambda^\vee_{\mathbb{C}}$ is differentiable and

$$\mathrm{Im}(Z'_t(E) \cdot \bar{Z}_t(E)) \geq 0$$

for all $t \in [0,1]$ and σ_t-stable object $E \in \mathcal{D}$. Then for any object $0 \neq E \in \mathcal{D}$, the functions

$$t \to \phi^-_t(E), t \to \phi^+_t(E)$$

are weakly-increasing.

To get inequality

$$\phi^-_1(F) \geq \phi_0(F)$$

Sketch of proof

However, we know that $\otimes L$ will map semistable object of σ_1 to σ_0. And we have

$$\phi_1(F) \geq \phi_0(F) > \phi_0(E) = \phi_1(E \otimes L^{-1})$$

And therefore

$$\mathrm{Hom}(F \otimes L, E) \cong \mathrm{Hom}(F, E \otimes L^{-1}) = 0$$

Outline

Further questions

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3. Can we show that these are all the stability condition with global dimension 3 and then use the same procedure in $\mathbb{P}^2$ case to show the contractibility of $\mathrm{Stab}(\mathbb{P}^3)$ and/or other varieties?

Thank you for listening!