Notes for Homological Mirror Symmetry

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Contents

1 Introduction to homological mirror symmetry (2022-02-27 Su Weilin)

1.1 GW invariants

Let (M, ω, J) be a symplectic manifold with symplectic form ω and compatible almost complex structure J. Gromov-Witten invariants are roughly the number

$$\#\{(\Sigma, u) \mid u : \Sigma \to M \text{ (pseudo-)holomorphic + constraints }\}$$

These invariants are introduced by Gromov around 1985, who proves that the zero dimensional part of above moduli space is finite. In mirror symmetry, Gromov-Witten invariants belong to A-model.

Example 1.1.1

$$\#\{\deg 1 \text{ } curves \text{ } u : \mathbb{CP}^1 \to \mathbb{CP}^n \text{ } passing \text{ } through \text{ } 2 \text{ } generic \text{ } points\} = 1$$

However, in general the Gromov-Witten invariants are very difficult to compute because the moduli space of (pseudo-)holomorphic curves is far from smooth and intersections are not transversal. So people want to find some indirect ways to compute these invariants.

1.2 Mirror symmetry

Now consider (M, Ω) which is a complex manifold where Ω is the complex structure. We can consider the sheaf cohomology and peroid integration of differential forms. Period integral belongs to B-models.

Suggested by physicists, there exists a diagram

$$
\begin{array}{ccc}
\text{GW invariants} & \longleftarrow & \text{symplectic manifold } M \\
\Big\updownarrow & & \Big\updownarrow{\scriptstyle\text{Mirror}} \\
\text{period integral} & \longleftarrow & \text{complex manifold } M^\vee
\end{array}
$$

where M and $M^\vee$ are called *mirror dual*. The problem of computing GW invariants can be transformed to the calculation of period integral. We now go to consider Kähler manifold (M, ω, Ω), where ω is symplectic form

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and Ω is complex structure where symplectic form and complex structure are compatible. Physically, above correspondence is from the duality of 2 dimensional supersymmetric field theory and is checked for quintic 3-fold. The question is why they coincide mathematically?

1.3 Homological mirror symmetry

The idea is to replace the space by some kind of categories. The correct "category" is the A_∞-category, introduced by Stasheff in 1963 to study group like topological spaces.

Definition 1.3.1 *An A_∞-category is following collection of data.*

1. *A set of objects.*

2. *Morphisms between objects are $\mathbb{Z}$-graded linear space $\hom(X, Y)$.*

3. *m_k the "composition" of morphisms*

$$m_k : \hom(X_0, X_1) \otimes \cdots \otimes \hom(X_{k-1}, X_k) \to \hom(X_0, X_k)$$

satisfying the A_∞-relation

$$\sum_{i,j} (-1)^{\sum_1^i |x_l| + 1} m_{k+1-j}(x_1, \cdots, x_i, m_j(x_{i+1}, \ldots, x_{i+j}), x_{i+j+1}, \ldots, x_k) = 0$$

here $|x_i|$ denote the grading of x_i.

for exmple, when $k = 1$, we have $m_1(m_1(x)) = 0$ that is $m_1^2 = 0$ is a differential. For $k = 2$, the A_∞-relation gives Leibniz rule. It is more convenient to consider on homology level by differential m_1. That is $\mathrm{Hom}^*(X, Y) = H^*(\hom(X, Y), m_1)$ and define the composition to be

$$\mathrm{Hom}^*(X, Y) \otimes \mathrm{Hom}^*(Y, Z) \to \mathrm{Hom}^*(X, Z)$$
$$[x] \otimes [y] \to [x] \cdot [y] := (-1)^{|x|}[m_2(x, y)]$$

In above notation, the $k = 3$ relation gives associativity $([x] \cdot [y]) \cdot [z] = [x] \cdot ([y] \cdot [z])$.

Let $(M^\vee, \Omega)$ be a complex manifold, we can view it as an algebraic variety and consider the derived category of coherent sheaves of the variety which can be enhanced to be a dg-category. A rough definition of derived category is that the objects are bounded complexes of coherent sheaves and morphisms are $\mathrm{Ext}^*(E, F)$ and composition conditions. It is obtained from category of complexes by formally inverting the quasi-isomorphisms with some additional universal properties. This is the story on the complex side.

The story on the symplectic side is Fukaya category. The objects of Fukaya category is $\{L \subset M \mid L \text{ (compact) Lagrangian }\}$. The morphism is generated by intersection points, that is

$$\hom(L_1, L_2) = R\langle L_1 \cap L_2 \rangle$$

for transveral intersections. A Fukaya category is a A_∞ category with coefficients of the composition map couting holomorphic discs satisfying some relations. Kontsevich suggest the two categories are related.

Conjecture 1.3.2 *[Homological Mirror Symmetry, [Kon95]] For any Calabi-Yau M there exists a mirror dual $M^\vee$ such that*

$$\mathrm{Fuk}(M^\vee) \cong \mathrm{D}^b(\mathrm{Coh}M) \quad \mathrm{Fuk}(M) \cong \mathrm{D}^b(\mathrm{Coh}M^\vee)$$

The above diagram is completed to be the following

$$
\begin{array}{ccccc}
\text{GW invariants} & \longleftarrow & \text{symplectic manifold } M & \longrightarrow & \mathrm{Fuk}(M) \\
\Big\updownarrow & & \Big\updownarrow{\scriptstyle\text{Mirror}} & & \Big\updownarrow{\scriptstyle\text{HMS}} \\
\text{period integral} & \longleftarrow & \text{complex manifold } M^\vee & \longrightarrow & D^b(\mathrm{Coh}M^\vee)
\end{array}
$$

To go from period integral to Gromov-Witten invariants, we want to get some information of symplectic manifold M from its Fukaya category. We consider Hochschild cohomology. For associative algebras, we define

Definition 1.3.3 *We define **Hochschild complex** (HC_*, b) for a k-algebra to be*

$$\mathrm{HC}_p(A) = A^{\otimes(p+1)}$$

$$d_i : a_0 \otimes \cdots \otimes a_p \to a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p, \quad (i = 0, \cdots, p-1)$$

$$d_p : a_0 \otimes \cdots \otimes a_p \to a_p a_0 \otimes \cdots \otimes a_{p-1}$$

$$b = \sum_i (-1)^i d_i$$

*Then we define the homology of above complex to be **Hochschild homology** and cohomology of dual complex to be **Hochshild cohomology**, denoted by HH_* and HH^* respectively.*

We can extend above definition to A_∞-categories. Following conjecture relates Fukaya category to the geometry of original symplectic manifold.

Conjecture 1.3.4 *[Kontsevich?]*

$$H^*(M) \cong \mathrm{HH}^*(\mathrm{Fuk(M)})$$

2 Derived category and triangulated category (2022-03-06 Zhang Nantao)

2.1 References

Chapter 1 of [KS94], Chapter 1 and 2 of [Huy06], Chapter 3 of [GM03].

2.2 Derived category

First, recall the complex of an abelian category $\mathcal{A}$ is of the form

$$A^* : \cdots \to A^{n-2} \xrightarrow{d^{i-1}} A^{n-2} \xrightarrow{d^i} A^n \to \cdots$$

satisfying $d^i \circ d^{i-1} = 0$. Morphism between the complex A^* and B^* are a series of morphisms $f^i : A^i \to B^i$ making the following diagram commutes.

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f^{n-2}} & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & \\
\cdots \longrightarrow & B^{n-2} & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow \cdots
\end{array}
$$

where the differentials are omitted.

We then have a category of complex denoted by $Kom(\mathcal{A})$ where objects are complexes of $\mathcal{A}$ and morphisms are given as above.

There exists a natural functor called **shift functor**, $T : Kom(\mathcal{A}) \to Kom(\mathcal{A})$, such that

$$(T(A^*))^i := A^{i+1}$$

$$\mathrm{d}^*_{T(A^*)} := -\,\mathrm{d}_A^{i+1}$$

For $f^* : A^* \to B^*$, we have

$$T(f^*) = f^{i+1}$$

Obviously, T is an equivalence of category. Usually, we denoted $T(A)$ by $A[1]$ and $T(f)$ by $f[1]$, and we use $A[n]$ and $A[-1]$ in an obvious way.

Definition 2.2.1 *Recall, the ith **cohomology** of A^* denoted by $H^i(A^*) := \frac{\ker(\mathrm{d}^i)}{\mathrm{im}(\mathrm{d}^{i-1})} \in \mathcal{A}$.*

*A^* is called **acyclic** if $H^i(A^*) = 0$ for all $i \in \mathbb{Z}$.*

*$f^* : A^* \to B^*$ induces morphisms $H^i(f) : H^i(A) \to H^i(B)$ if all induced morphism are isomorphisms then we call f a **quasi-isomorphism** (qis for short).*

Remark 2.2.2 *There exists complexes with same cohomology group but not quasi-isomorphic. For example*

$$\mathbb{C}[x,y]^{\oplus 2} \xrightarrow{(x,y)} \mathbb{C}[x,y]$$

$$\mathbb{C}[x,y] \xrightarrow{0} \mathbb{C}$$

We first give a definition of derived category by universal properties.

Definition 2.2.3 *The **derived category** of $\mathcal{A}$ is a category $D(\mathcal{A})$ with a functor $Q : Kom(\mathcal{A}) \to D(\mathcal{A})$, such that*

1. *If $A^* \to B^*$ qis in $Kom(\mathcal{A})$ then $Q(f)$ is an isomorphism in $D(\mathcal{A})$.*

2. *Any functor $F : Kom(\mathcal{A}) \to D$ satisfying condition (1) uniquely factor through Q. That is there exists unique G making the following diagram commutes*

Before giving a construction of derived categories, we notice that the cohomology is well defined in derived category and $\mathcal{A} \to Kom(\mathcal{A}) \to D(\mathcal{A})$ is a full subcategory.

Definition 2.2.4 *Given an abelian category $\mathcal{A}$, we define **homotopy category** $K(\mathcal{A})$ to be following data*

$$\mathrm{Ob}(K(\mathcal{A})) := \mathrm{Ob}(Kom(\mathcal{A}))$$

$$\mathrm{Hom}_{K(\mathcal{A})}(A^*, B^*) := \mathrm{Hom}_{Kom(\mathcal{A})}(A^*, B^*)/\sim$$

where $\sim$ denote the homotopy equivalence.

Recall two morphism of complexes $f, g : A^* \to B^*$ are called **homotopy equivalent** if there exists a collection of homomorphisms $h^i : A^i \to B^{i-1}$ such that $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.

Notice that if $f \circ g \sim \mathrm{id}$ and $g \circ f \sim \mathrm{id}$, then f and g are all quasi-isomorphisms.

Now we give another definition of derived category.

Definition 2.2.5 *A **derived category** is the following collection of data*

$$\mathrm{Ob}(D(\mathcal{A})) := \mathrm{Ob}(Kom(\mathcal{A}))$$

two moprhisms are equivalent if there exists following commutative diagram in $K(\mathcal{A})$.

The composition of morphisms are given by

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The associativity of the composition is obvious. To check that $D(\mathcal{A})$ is indeed a category, we only need to check that

1. C_0^* exists.

2. The composition is unique.

To address the above two questions, we introduce the notion of mapping cone.

Definition 2.2.6 *Let $f^* : A^* \to B^*$ we define the **mapping cone** $C(f)^i := A^{i+1} \oplus B^i$ and*

$$\mathrm{d}_{C(f)}^* := \begin{pmatrix} -\mathrm{d}_A^{i+1} & 0 \\ f^{i+1} & \mathrm{d}_B^i \end{pmatrix}$$

Given a morphism $f : A^* \to B^*$, we have long exact sequence

$$\to H^i(A^*) \to H^i(B^*) \to H^i(C(f)) \to H^{i+1}(A^*) \to \dots$$

Proposition 2.2.7 *Let $f^* : A^* \to B^*$ a morphism, $C(f)$ its mapping cone, and diagram of solid arrows*

$$
\begin{array}{ccccccc}
B^* & \xrightarrow{\tau} & C(f) & \xrightarrow{\pi} & A^*[1] & \xrightarrow{-f} & B^*[1] \\
\downarrow{=} & & \downarrow{=} & & \downarrow{g} & & \downarrow{=} \\
B^* & \xrightarrow{\tau} & C(f) & \xrightarrow{\tau_\tau} & C(\tau) & \xrightarrow{\pi'} & B^*[1]
\end{array}
$$

Then there eixsts an isomorphism in $K(\mathcal{A})$, $g : A^*[1] \to C(\tau)$ making the diagram commutes in $K(\mathcal{A})$.

PROOF Let $g = (-f^{i+1}, \mathrm{id}, 0)$ and check the commutativity.

Remark 2.2.8 *The above isomorphism exists in $K(\mathcal{A})$ but not in $Kom(\mathcal{A})$ so we need to start from homotopy category instead of category of complexes.*

Proposition 2.2.9 *Given a diagram*

$$
\begin{array}{ccc}
 & & C^* \\
 & & \downarrow{g} \\
A^* & \xrightarrow{f \ qis} & B^*
\end{array}
$$

there exists C_0^* fill the following diagram

$$
\begin{array}{ccc}
C_0^* & \xrightarrow{qis} & C^* \\
\downarrow & & \downarrow{g} \\
A^* & \xrightarrow{f \ qis} & B^*
\end{array}
$$

PROOF We fill the diagram gradually, first we have

$$
\begin{array}{ccccccc}
 & & C^* & & & & \\
 & & \downarrow{g} & & & & \\
A^* & \xrightarrow{f} & B^* & \xrightarrow{\tau} & C(f) & \longrightarrow & A^*[1]
\end{array}
$$

By Theorem 2.2.7, we can fill to the following diagram by isomorphism $A^*[1] \cong C(\tau \circ g)$.

$$
\begin{array}{ccccccc}
C^* & \xrightarrow{\tau \circ g} & C(f) & \longrightarrow & C(\tau \circ g) \\
\downarrow{g} & & \downarrow & & \downarrow \\
A^* & \xrightarrow{f} & B^* & \xrightarrow{\tau} & C(f) & \longrightarrow & A^*[1]
\end{array}
$$

And finally,

$$
\begin{array}{ccccccc}
C(\tau \circ g)[-1] & \longrightarrow & C^* & \xrightarrow{\ \tau \circ g\ } & C(f) & \longrightarrow & C(\tau \circ g) \\
\downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow \\
A^* & \xrightarrow{\ f\ } & B^* & \xrightarrow{\ \tau\ } & C(f) & \longrightarrow & A^*[1]
\end{array}
$$

where $C(\tau \circ g)[-1]$ is the required C_0^* in the proposition. We only need to show that $C(\tau \circ g)[-1] \to C^*$ is a quasi-isomoprhism, but it follows from long exact sequence and the five lemma.

We now solve the existence of the composition, but the uniqueness problem is still left.

Definition 2.2.10 *A class of morphism $S \subset \mathrm{Mor}(\mathcal{A})$ is said to be **localizing** if*

1. *S is closed under compositions and $\mathrm{id}_X \in S$ for every $S \in \mathrm{Ob}(\mathcal{A})$.*

2. *Excision condition, that is for any $f \in \mathrm{Mor}(\mathcal{A})$ and $s \in S$, there exists $g \in \mathrm{Mor}(\mathcal{A})$ and $t \in S$ such that*

$$
\begin{array}{ccc}
C_0^* & \xrightarrow{\ g\ } & C^* \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\
A^* & \xrightarrow{\ f\ } & B^*
\end{array}
$$

   is commutative.

3. *Let $f, g \in \mathrm{Hom}(X, Y)$, the existence of $s \in S$ such that $sf = sg$ is equivalent to the existence of $t \in S$ with $ft = gt$.*

Remark 2.2.11 *Quasi-isomorphisms don't form a localizing class in $Kom(\mathcal{A})$ but in $K(\mathcal{A})$.*

Only condition 3 need to be checked for quasi-isomorphism class. Given $f^* : A^* \to B^*$ in $K(\mathcal{A})$ and a quasi-isomorphism $s : B^* \to \bar{B}^*$ with $sf = 0$, we want to show that there exists $t : \bar{A}^* \to A^*$ with $ft = 0$. To see that we only need to see the following diagram

$$
\begin{array}{ccccc}
C(s)[-1] & \xrightarrow{\ \tau[-1]\ } & B^* & \xrightarrow{\ s\ } & \bar{B}^* \\
\downarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle f} & & \\
C(s)[-1] & \xleftarrow{\ g\ } & A^* & \xleftarrow{\ t\ } & C(g)[-1]
\end{array}
$$

where $g^i : A^* \to B^i \oplus \bar{B}^{i-1}$ is the map

$$
g^i : (a^i) \to (f^i(a^i), -h^i(a^i))
$$

where $h^i : A^i \to \bar{B}^{i-1}$ is the homotopy bewteen sf and 0. Then $ft = \tau[-1]gt = 0$ and t is quasi-isomorphism by long exact sequence.

Remark 2.2.12 *The derived category is additive but not ableian, as it does not always have kernels and cokernels. But it is a triangulated category, which we will introduce later.*

2.3 Triangulated category

Definition 2.3.1 *A **triangulated category** is an additve category D with an additive functor $T : D \to D$ called the **shift functor** and a set of **distinguished triangles** satisfying 4 axioms. In convention, we use notation $A[1] := T(A)$ and $f[1] := T(f)$.*

1. *(a) Any triangle of the form*

$$
A \xrightarrow{\ \mathrm{id}\ } A \to 0 \to A[1]
$$

   is distinguished.

   (b) Any triangle isomorphic to a distinguished triangle is distinguished.

(c) Any morphism $f : A \to B$ can be completed to a distinguised triangle

$$A \xrightarrow{f} B \to C \to A[1]$$

2.

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguised triangle if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1]$$

is.

3. If there exists a commutative diagram of solid arrows of the following diagram,

$$
\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \vdots & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1]
\end{array}
$$

then there exists a dahsed arrows $h : C \to C''$ (not necessarily unique) to complete the diagram.

4. Suppose given distinguished triangles

$$X \xrightarrow{f} Y \to Z' \to X[1]$$

$$Y \xrightarrow{g} Z \to X' \to Y[1]$$

$$X \xrightarrow{g \circ f} Z \to Y' \to X[1]$$

then there exists a distinguished triangle

$$Z' \to Y' \to X' \to Z'[1]$$

such that the following diagram is commutative

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & X[1] \\
\downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_X[1]} \\
X & \xrightarrow{g \circ f} & Z & \longrightarrow & Y' & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_Z} & & \downarrow & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & Y[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z'[1]
\end{array}
$$

Remark 2.3.2 *Basically, the 4th axiom means that $(A/C)/(B/C) = A/B$ in abstract algebra. That axiom is sometimes called octahedral axiom because it can be arranged into an octahedral.*

Proposition 2.3.3 *A derived category is a triangulated category where the distinguished triangles are those triangles isomorphic to*

$$A \xrightarrow{f} B \to C(f) \to A[1]$$

Remark 2.3.4 *It may be interesting to know other examples of triangulated categories. One example is the derived category of A_∞-category and another may be given by stable homotopy category. The basic idea is from the cofiber sequence in topology.*

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y \to \cdots$$

and making $T := \Sigma$. However, Σ is not invertible, so we have to do some additional work to make category of topological space a triangulated category.

Lemma 2.3.5 *For distinguished triangle*

$$A \to B \to C \to A[1]$$

the composition $A \to C = 0$.

PROOF Consider the following diagram

$$
\begin{array}{ccccccc}
A & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \vdots & & \downarrow \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1]
\end{array}
$$

Proposition 2.3.6 *Let $A \to B \to C \to A[1]$ be a distinguished triangle, and $A_0 \in D$, then*

$$\mathrm{Hom}(A_0, A) \to \mathrm{Hom}(A_0, B) \to \mathrm{Hom}(A_0, C)$$

$$\mathrm{Hom}(C, A_0) \to \mathrm{Hom}(B, A_0) \to \mathrm{Hom}(A, A_0)$$

are exact.

PROOF By Theorem 2.3.5, we have the composition of maps are equal to 0. To show that it is exact, we only need to consider the following diagram

$$
\begin{array}{ccccccc}
A_0 & \longrightarrow & A_0 & \longrightarrow & 0 & \longrightarrow & A_0[1] \\
\vdots & & \downarrow & & \downarrow & & \vdots \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1]
\end{array}
$$

Definition 2.3.7 *An additive functor $F : D \to D'$ between triangulated categories D and D' is called **exact** if the following two conditionsare satisfied:*

1. *There exists a functor isomorphism*

$$F \circ T_D \cong T_{D'} \circ F$$

2. *Any distinguished triangle in D is mapped to distinguished triangle in D'.*

Proposition 2.3.8 *Let $F : D \to D'$ be an exact functor. If $F \dashv H$, then $H : D' \to D$ is exact. Similar result holds for $G \dashv F$.*

PROOF We first check the commutativity with shift functor.

$$
\begin{aligned}
\mathrm{Hom}(A, H(T'(B))) &\cong \mathrm{Hom}(F(A), T'(B)) \\
&\cong \mathrm{Hom}(T'^{-1}(F(A)), B) \\
&\cong \mathrm{Hom}(F(T^{-1}(A)), B) \\
&\cong \mathrm{Hom}(T^{-1}(A), H(B)) \\
&\quad \mathrm{Hom}(A, T(H(B)))
\end{aligned}
$$

Then we check that H maps distinguished triangles to distinguished triangles. Let $A \to B \to C \to A[1]$ distinguished in D'. We can completed to a distinguished triangle

$$H(A) \to H(B) \to C_0 \to H(A)[1]$$

in D. By applying F, and use adjoint property and exactness of F, we have

$$
\begin{array}{ccccccc}
F(H(A)) & \longrightarrow & F(H(B)) & \longrightarrow & F(C_0) & \longrightarrow & F(H(A))[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1]
\end{array}
$$

Applying H, combine two diagram and using adjointness $h : \mathrm{id} \to H \circ F$, we have

$$
\begin{array}{ccccccc}
HA & \longrightarrow & HB & \longrightarrow & C_0 & \longrightarrow & HA[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
HFHA & \longrightarrow & HFHB & \longrightarrow & HFC_0 & \longrightarrow & HFHA[1] \\
\downarrow & & \downarrow & & {\scriptstyle\xi}\downarrow & & \downarrow \\
HA & \longrightarrow & HB & \longrightarrow & HC & \longrightarrow & HA[1]
\end{array}
$$

The curved morphisms are isomorphism. And by exact sequence and five lemma, we have

$$\mathrm{Hom}(A_0, C_0) \cong \mathrm{Hom}(A_0, H(C))$$

for all A_0 and hence

$$\xi : C_0 \cong H(C)$$

is an isomorphism. And therefore $H(A) \to H(B) \to H(C) \to H(A)[1]$ is isomorphic to $H(A) \to H(B) \to C_0 \to H(A)[1]$ and is therefore distinguished.

Definition 2.3.9 *Two triangulated categories D and D' are **equivalent** if there exists an exact equivalence $F : D \to D'$.*

*If D is a triangulated category the set $\mathrm{Aut}(D)$ of isomorphism classes of equivalence $F : D \to D$ forms the **group of autoequivalence**.*

*A subcategory $D' \subset D$ of a triangulated category is a **triangulated subcategory** if D' admits the structure of triangulated category such that the inclusion $D' \hookrightarrow D$ is exact.*

We say a set of objects S **generates** triangulated category D if any triangulated subcategory of D contains S is D itself.

3 Derived functors and some examples of derived category of coherent sheaves (2022-03-13 Zhang Nantao)

3.1 References

Chapter 1 of [KS94], Chapter 1 and 2 of [Huy06], Chapter 3 of [GM03].

3.2 Derived functors

To define the derived functor, we first give a definition of boundedness.

Definition 3.2.1 *Let $Kom^*(\mathcal{A})$ with $* = +, -$ or b be the category of complexes A^* with $A^i = 0$ for $i \ll 0$, $i \gg 0$, $|i| \gg 0$, and called **bounded from below**, **bounded from above** and **bounded**. We can define subcategory $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ similarly.*

Proposition 3.2.2 *The natural functor $D^*(\mathcal{A}) \to D(\mathcal{A})$ defines equivalence of $D^*(A)$ with the full triangulated subcategories of all complexes $A^* \in D(\mathcal{A})$ with $H^i(A^*) = 0$ for $i \ll 0$, $i \gg 0$, $|i| \gg 0$.*

Remark 3.2.3 *The above proposition is not true, if we replace $D(\mathcal{A})$ by $K(\mathcal{A})$.*

We now give a formal definition of derived functors.

Lemma 3.2.4 *Let $F : K^*(\mathcal{A}) \to K^*(B)$ be an exact functor of triangulated categories. Then F naturally induces a commutative diagra*

$$
\begin{array}{ccc}
K^+(\mathcal{A}) & \longrightarrow & K^+(\mathcal{B}) \\
\downarrow & & \downarrow \\
D^+(\mathcal{A}) & \longrightarrow & D^+(\mathcal{B})
\end{array}
$$

if one the following two conditions holds (in fact two conditions are equivalent)

1. F maps quasi-isomorphism to quasi-isomorphism.

2. F maps acyclic complex to acyclic complex.

For $F : \mathcal{A} \to \mathcal{B}$ left exact, we have $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ satisfy above lemma and therefore induces a derived functor.

$$
RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})
$$

Dually, for right exact functor we have

$$
LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})
$$

The construction above is quite formal, we reuiqre some more explicit process for computation. Recall an abelian category contains enough injective if for any object $A \in \mathcal{A}$ there exists an injective morphism $A \to I$ with $I \in \mathcal{A}$ injective.

Proposition 3.2.5 *Supose A is an abelian category with enough injectives. For any $A^* \in K^+(\mathcal{A})$, there exists a complex $I^* \in K^+(\mathcal{A})$ with $I^i \in \mathcal{A}$ injective objects and quasi-isomorphism $A^* \to I^*$.*

PROOF We construct the injective resolution directly. We may assume that first nonzero element in A^* is A^0 or we shifted A to make it so. at position 0, we consider

And if we already have I^i, we have step $i + 1$ by following construction.

Then you may check it is indeed a quasi-isomorphism. For details, you may consult [GM03].

Lemma 3.2.6 *Suppose $A^* \to B^*$ quuasi-isomorphism between two complexes in $K^+(\mathcal{A})$ then for any complex I^* of injective objects I^i with $I^i = 0$ for $i \ll 0$ the induced map*

$$
\operatorname{Hom}_{K(\mathcal{A})}(B^*, I^*) \cong \operatorname{Hom}_{K(\mathcal{A})}(A^*, I^*)
$$

is bijective.

PROOF By distinguished triangles and long exact sequence, we only need to prove that for acyclic C^*, we have $\mathrm{Hom}(C^*, I^*) = 0$. Let $g \in \mathrm{Hom}(C^*, I^*)$, we show that it is homotopic to 0 map. We argue by induction, first, for small enough i, we have $C^i = I^i = 0$, which may serve as the start of the induction. If we have h^j for $j \leq i$. Then we have $g^i - d_I^{i-1} \circ h^i : C^i \to I^i$ factor through C^i / C^{i-1} by acyclic property of C^i. Then by injectivity of I^i, we may lift it to $h^{i+1} : C^{i+1} \to I^i$ such that $g^i - d_I^{i-1} \circ h^i = h^{i+1} \circ d_C^i$.

Lemma 3.2.7 *Let $A^*, I^* \in Kom^+(\mathcal{A})$ such that all I^i are injective. Then*

$$\mathrm{Hom}_{K(\mathcal{A})}(A^*, I^*) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^*, I^*)$$

PROOF

For any roof consisting of solid lines, it is equivalent to a dashed line representing a morphism in $\mathrm{Hom}(A^*, I^*)$.

Proposition 3.2.8 *If $\mathcal{A}$ is an abelian category with enough injectives. Then the functor*

$$\iota : K^+(\mathcal{I}) \to D^+(\mathcal{A})$$

is an equivalence. $\mathcal{I}$ is the full subcategory of all injectives of $\mathcal{A}$.

Now we come back to derived functor. We have another definition for derived functor. Consider the diagram

And we define $RF := Q_B \circ K(F) \circ \iota^{-1}$ is a well defined **derived functor**.

And we define
$$R^i F(A^*) := H^i(RF(A^*))$$

And object $A \in \mathcal{A}$ is called F-**acyclic** if $R^i F(A) = 0$ for $i \neq 0$.

Remark 3.2.9 *We can develop the dual theory for left exact functor and $D^-(\mathcal{A})$ with $\mathcal{A}$ having enough projectives. However, it is not so useful in algebraic geometry because category of coherent sheaves may not have enough projectives! [Har08, Chapter III]*

By above remark, we will need a more general framework to do derived functors.

Definition 3.2.10 *Let $F : \mathcal{A} \to \mathcal{B}$. A class of objects $\mathcal{I}_F \subset \mathcal{A}$ stable under finite sum is F-**adpated** if the following conditions hold:*

1. *If $A^* \in K^+(\mathcal{A})$ acyclic with $A^* \in \mathcal{I}_F$ for all i, then $F(A^*)$ is acyclic.*

2. *Any object in A can be embedded into an object of ci_F.*

Let $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$. We define triangulated subcategory $K_F \subset K^+(\mathcal{A})$ F-**adpated** if it satisfying following conditions.

1. *If $A^* \in K_F$ is acyclic, then $F(A^*)$ is.*

2. *Any $A^* \in K^+(A)$ is quasi-isomorphic to a complex in K_F.*

If $\mathcal{I}_F$ is F-adapted, then $K^+(\mathcal{I}_F)$ is F-adapted.

Proposition 3.2.11 *Supose A^*, B^* abelian category and $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ exact functor, and there exists an F-adpated class K_F. Then there exists a right derived functor RF satisfying*

1. *The following diagram commutes.*

$$
\begin{array}{ccc}
K^+(\mathcal{A}) & \xrightarrow{K(F)} & K^+(\mathcal{B}) \\
\downarrow {\scriptstyle Q_\mathcal{A}} & & \downarrow {\scriptstyle Q_\mathcal{B}} \\
K^+(\mathcal{A}) & \xrightarrow{RF} & K^+(\mathcal{B})
\end{array}
$$

2. *(Universal property) Suppose $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is exact. Then $Q_B \circ K(F) \to G \circ Q_A$ factor through a unique morphism*

$$RF \to G$$

Therefore, we can use flat resolution or other F-acyclic resolutions to do the computations.

We can check that

$$\mathrm{Ext}^i(A, -) := H^i \circ R\,\mathrm{Hom}(A, -)$$

For $A, B \in \mathcal{A}$ view as complex concentrated in degree 0.

Also, we have $\mathrm{Hom}^* : K^+(\mathcal{A})^{op} \times K^+(\mathcal{A}) \to K(\mathcal{A})$ defined by

$$\mathrm{Hom}^i(A^*, B^*) := \oplus \mathrm{Hom}(A^k, A^{k+i})$$

$$d(f) := d_B \circ f - (-1)^i f \circ d_A$$

And

$$\mathrm{Ext}^i(A^*, B^*) := H^i(R\,\mathrm{Hom}^*(A^*, B^*))$$

By above definition, we have

$$\mathrm{Ext}^i(A^*, B^*) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^*, B^*[i])$$

Proposition 3.2.12 *Let $F_1 : \mathcal{A} \to \mathcal{B}$ and $F_2 : \mathcal{B} \to \mathcal{C}$ left exact functor and adapted class $\mathcal{I}_{F_1} \subset \mathcal{A}$, $\mathcal{I}_{F_2} \subset \mathcal{B}$ such that $F_1(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$, then there is a natural transformation*

$$R(F_2 \circ F_1) \cong RF_2 \circ RF_1$$

3.3   Some results about coherent sheaves

We have following comparison between complex algebraic geometry and complex analytic geometry.

Complex algebraic geometry	complex analytic geometry
scheme / variety	complex analytic space
affine scheme	$\{f(z^i) = 0\}$
regular function	holomorphic function
morphism	holomorphic morphism
locally free sheaves	vector bundles
Zariski topology	analytic topology

We now introduce two famous result to communicates betwee complex algebraic geometry and complex analytic geometry.

Theorem 3.3.1 *[Serre's GAGA [Ser56]]   Given algebraic variety X, we have*

$$X \to X^{an}$$

making X an analytic space. Moreover the coherent sheaves on X maps to coherent sheaves on X bijectively.

Theorem 3.3.2 *[Chow's lemma [Cho49]]   A compact analytic variety in $\mathbb{P}^n$ is an algebraic variety.*

So for projective variety, we can freely exchange the view of complex algebraic geometry and complex analytic geometry. By Jocobian crieteria, the irreducible smooth projective complex algebraic variety is a complex *manifold.*

By [Har08, Ex II.5.18, Ex III.6.8, 6.9], we have following result.

Proposition 3.3.3 *We have one to one correspondence between locally free sheaves of rank n on Y and isomorphism classes of vector bundles of rank n over Y.*

Proposition 3.3.4 *If X is Noetherian (for example, projective or affine), integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of locally free sheaf. Moreover, the locally free resolution is of finite length.*

As regular schemes are locally factorial and regular is equivalent to smooth in characteristic 0. We can work in complex manifold with vector bundles if you wish to. Last, we rewrite Serre duality in the language of derived categories. We use $D(X)$ to denote derived category of coherent sheaves on X.

Theorem 3.3.5 *[Serre duality]* *We define*

$$S_X : D^b(X) \to D^b(X)$$
$$F^* \to (\omega_X \otimes^L F^*)[n]$$

where ω_X is the dualizing sheaf and $n = \dim X$. Then we have

$$\mathrm{Hom}(E^*, F^*) \cong \mathrm{Hom}(F^*, S_X(E))^\vee$$

Remark 3.3.6 *The category with a Serre functor equal to shifting is called a **Calabi-Yau category**. (For general definition of Serre functor, see [Huy06]) In [Kon95], Kontsevich make Calabi-Yau property as a sign for equivalence between derived category of coherent sheaves and derived category of Fukaya category.*

3.4 Examples of derived category of coherent sheaves

We now consider two examples of schems. First, we consider $X = \mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[x]$. Two coherent sheave $\mathscr{O}_X$ trivial line bundle and skyscrapper sheaf $\mathscr{O}_a, a \in \mathbb{A}^1$ generates the category $\mathrm{Coh}(X)$ and therefore $D^b(X)$. We have $\mathrm{Coh}(\mathrm{Spec}\,A) = A$-mod. The morphism between generators are all easy to compute. For example, we compute $R\,\mathrm{Hom}(\mathscr{O}_a, \mathscr{O}_b)$, we need to take projective resolution for $\mathscr{O}_a$, that is

$$0 \to \mathbb{C}[x] \xrightarrow{\times(x-a)} \mathbb{C}[x] \to \mathbb{C}[x]/(x-a) \to 0$$

which is an example of locally free resolution. Then we have

$$R\,\mathrm{Hom}(\mathscr{O}_a, \mathscr{O}_b) = \begin{cases} 0 & \text{if } a \neq b \\ \mathscr{O}_a \xrightarrow{0} \mathscr{O}_a & \text{if } a = b \end{cases}$$

the second sequence started from index 0.

A similar result holds for cylinder $S^1 \times \mathbb{R} \cong_{top} \mathbb{C}^* = \mathrm{Spec}\,\mathbb{C}[x, x^{-1}]$.

Now we consider another example $D^b(\mathbb{P}^1)$.

Proposition 3.4.1 *Let $M = \mathscr{O} \oplus \mathscr{O}(1)$. Then graded algebra $R\,\mathrm{Hom}(M, M)$ is concentrated in degree 0 and is the path algebra of Kronecker quiver $\bullet \rightrightarrows \bullet$.*

PROOF We have $\mathrm{Ext}^i(\mathscr{O}(l), \mathscr{O}(k)) = H^i(X, \mathscr{O}(k-l))$. For $i > 0$, this is nonzero unless $i = 1$ and $k - l = -2$ which is impossible. The equivalence between algebra is easy to see.

Definition 3.4.2 *A coherent sheaf T on X is called a **tilting sheaf** if*

1. *$A := \mathrm{End}_{\mathscr{O}_X}(T)$ has finite global dimension.*

2. *$\mathrm{Ext}^i_{\mathscr{O}_X}(T, T) = 0$ for $i > 0$.*

3. *T generates $D^b(X)$.*

Theorem 3.4.3 *Let T be a tilting sheaf on a smooth projective scheme X, with tilting algebra $A = \mathrm{End}_{\mathscr{O}_X}(T)$. Then the functors*

$$F(-) := \mathrm{Hom}_{\mathscr{O}_X}(T, -)$$

$$G(-) := - \otimes_A T$$

induces equivalence of triangulated categories

$$RF : D^b(X) \to D^b(A\text{-}mod)^{op}$$

$$LG : D^b(A\text{-}mod)^{op} \to D^b(X)$$

PROOF First, by smoothness of X, coherent sheaves has finite length resolution and by property (1) Theorem 3.4.2 A-mod^{op} has finite length resolution so we have RF, LG well defined as morphism between bounded categories.

By property (2) of Theorem 3.4.2, we have $RF \circ LG(A) = RF(T) = A$. Hence it is equivalence on finitely generated projective A-module and therefore on all A-modules. By property (3) of Theorem 3.4.2, we have the image of LG, the triangulated subcategory generated by T, is all of $D^b(X)$. So we for every F, we have $F = LG(M)$ for some M and therefore have $LG \circ RF(F) \cong LG \circ RF \circ LG(M) = LG(M) = F$.

Theorem 3.4.4 $\mathscr{O} \oplus \mathscr{O}(1) \oplus \cdots \oplus \mathscr{O}(n)$ *is a tilting sheaf on $X = \mathbb{P}^n$.*

4 Floer homology and Fukaya categories I (2022-03-27 Lan Zhuoming)

4.1 References

The main reference is [Aur13]. And for more details consult [Sei08].

4.2 Floer homology

Let (M, ω) symplectic manifold of dimension $2n$. Let J be ω-compactible almost complex structure. The set of ω-compatible almost complex structures

$$\mathcal{J}(M, \omega) = \{J \in \mathrm{End}(TM) \mid J^2 = -1, g_J = \omega(-, J-) \text{ is a Riemannian metric}\}$$

is contractible.

Example 4.2.1 *The cotagent bundle T^*S is a symplectic manifold with symplectic form $\omega = d\theta \wedge dr$ where θ is the coordinate of circle and r is the coordinate of cotangent fiber.*

Let L_0, L_1 Lagrangian submanifold of dimension n. By definition of Lagrangian manifold, $\omega|_{L_0} = 0, \omega|_{L_1} = 0$. For simplicity, we first assume that L_0, L_1 intersects transversally and L_0, L_1 compact and $[\omega] \cdot \pi_2(M, L_0) = 0$ and $[\omega] \cdot \pi_2(M, L_1) = 0$. Then for dimension reason $\chi(L_0, L_1) := L_0 \pitchfork L_1 = \{p_i\}$ is a finite set.

Definition 4.2.2 *Let k be a field. We define the **Novikov field** Λ as follows:*

$$\Lambda = \{\sum a_i T^{\lambda_i} \mid a_i \in k, \lambda_i \in \mathbb{R}^{\geq 0}, \lim_{i \to \infty} \lambda_i = +\infty\}$$

*The **Floer complex** $CF(L_0, L_1) = \Lambda\langle p_i \rangle$ freely generated module by $\{p_i\}$. The differential of the complex $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$.*

$$\partial(p) = \sum_{q \in \chi(L_0, L_1), \mathrm{ind}([u]) = 1} (\#\mathcal{M}(p, q, [u], J)) T^{w([u])} q$$

where $u \in \pi_2(M)$. The definition $\mathrm{ind}([u])$ and $\mathcal{M}(p, q, [u], J)$ will be discussed later.

If we want to count the points of moduli space $\mathcal{M}$ with sign, we need L_0, L_1 to be oriented with spin structure and char$k = 0$. Then $\mathcal{M}(p, q, [u], J)$ is orientable. If we count the points of moduli space $\mathcal{M}$ without sign, we require the base field char$k = 0$.

We count the number of u where

$$u : \mathbb{R} \times [0, 1] \to M^{2n}$$

satisfying following three conditions:

1. (Pseudo-holomorphic) Let $(u, s) \in \mathbb{R} \times [0, 1]$.

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0$$

2. (Boundary condition)

$$u(s, 0) \in L_0$$
$$u(s, 1) \in L_1$$
$$\lim_{s \to -\infty} u(s, t) = p$$
$$\lim_{s \to \infty} u(s, t) = q$$

3. (Finite energy condition)

$$E(u) = \int u^* \omega = \iint \left| \frac{\partial u}{\partial s} \right|^2 ds dt < +\infty$$

$$\widehat{\mathcal{M}}(p, q, [\pi], J) = \{u \mid u \text{ satisfies } (1)(2)(3) \text{ and } [u] = [\pi]\}$$

If $u(s, t)$ satisfies $(1)(2)(3)$, then $u(s + a, t)$ satisfies $(1)(2)(3)$. So we define

$$\mathcal{M}(p, q, [\pi], J) = \widehat{\mathcal{M}}(p, q, [\pi], J)/u(s + a, t) \sim u(s, t)$$

We now define **Maslov index**, $\text{ind}([u])$. Let $\text{LGr}(n)$ be Lagrangian Grassmanian of $\mathbb{R}^{2n} = \mathbb{C}^n$. Then we have $\pi_1(\text{LGr}(n)) = \mathbb{Z}$. There is the map

$$\overset{2}{\det} : \text{LGr}(n) \to S^1$$

which induces isomorphism on fundamental groups. To define Maslov index

$$u^* TM \cong \mathbb{R} \times [0, 1] \times \mathbb{R}^{2n}$$

$$u^* T_p L_0 = \mathbb{R}^n, \quad u^* T_p L_1 = i\mathbb{R}^n$$
$$u^* T_q L_0 = \mathbb{R}^n, \quad u^* T_q L_1 = i\mathbb{R}^n$$

Then $u^* T_p L_0 \to u^* T_p L_1 \to u^* T_q L_1 \to u^* T_q L_0 \to u^* T_p L_0$ gives a loop in $\text{LGr}(n)$. The element of fundamental group isomorphic to $\mathbb{Z}$ representing this loop is the Maslov index.

We now show that $\bar{\partial}_{J,u} = \frac{1}{2}(\frac{\partial}{\partial s} + J \frac{\partial}{\partial t})$. The differential of $\bar{\partial}_{J,u}$, $D_{\bar{\partial}_{J,u}}$, is a Fredholm operator (i.e. it has finite dimensional kernel and cokernel). We further require J to be regular, that is $D_{\bar{\partial}_{J,u}}$ is injective. By theory of analysis, we have

$$\dim \widehat{\mathcal{M}}(p, q, [u], J) = \text{Fredholm index} = \dim \text{coker} D_{\bar{\partial}_{J,u}} = \text{ind}([u])$$

By above equation, for $\text{ind}([u]) = 1$, we have

$$\dim \mathcal{M}(p, q, [u], J) = \dim \widehat{\mathcal{M}}(p, q, [u], J) - 1 = 0$$

So counting the points of $\dim \mathcal{M}(p, q, [u], J)$ is meaningful.

We now prove that $\partial^2 = 0$ for Floer complex. By Gromov compactness theorem, the limiting behavior of $\mathcal{M}(p, q, [u], J)$ will be broken strip or bubbling. By vanishing of second relative homotopy group the second case is impossible. So we have

$$\partial_{\text{ind}([u])=2} \bar{\mathcal{M}}(p, q, [u], J) = \coprod \mathcal{M}(p, r, [u'], J) \times \mathcal{M}(r, q, [u''], J)$$

The right hand side is the q coefficient in $\partial^2 p$. And by differential geometry theory, the counting of points of boundary of 1-dimensional manifolds is 0. So we have $\partial^2 = 0$.

4.3 Grading of Floer homology

We wish to give a grading on Floer complex such that $\text{ind}([u]) = \deg q - \deg p$. To give a $\mathbb{Z}$-grading of elements, we require:

1. $2c_1(TM) = 0$.

2. $\mu_L = 0 \in H_1(L, \mathbb{Z})$.

The condition $2c_1(TM) = 0$ gives a fiberwise universal covering $\widehat{\text{LGr}}(n) \to \text{LGr}(TM)$. For a loop $s : S^1 \subset L \to \text{LGr}(TM)$, we can lift it to a loop $\tilde{s} : S^1 \subset L \to \widehat{\text{LGr}}(n)$, which may not be a loop, so give a element of fundamental group of $\text{LGr}(n)$ isomorphic to $\mathbb{Z}$. So it defines an isomorphism $\text{Hom}(\pi(L), \mathbb{Z}) = H_1(L, \mathbb{Z})$ called **Maslov class** μ_L. If $\mu_L = 0$, then s can always be lifted to a loop.

Therefore, if two conditions are all satisfied, then given p, q two points, the difference of degree of p and q is given by lifting the section s_{L_1} linking p, q and counting . By above, lifting of s_{L_2} will give the same answer. So the difference of degrees are well defined.

4.4 Discussion of non-transversal case

If (L_0, L_1) don't intersect transversally and J is not regular, then we can always find time dependent almost complex structure $J(t)$ and Hamiltonian $H(t)$. Let ϕ_H^t is a Hamiltonian flow. And we define $CF_0(L_0, L_1, J(t), H(t))$, such that $L_0, (\phi_H^1)^{-1}L_1$ intersects transversally and $(\phi_H^t)_*^{-1}J(t)$ regular for $t > 0$. The pseudo-holomorphic conditions now comes

$$\frac{\partial \tilde{u}}{\partial s} + \tilde{J}(t)\frac{\partial \tilde{u}}{\partial t} = 0$$

$$\tilde{J}(t) = (\phi_H^t)_*^{-1}J(t)$$

$$\tilde{u}(s, t) = (\phi_H^t)^{-1}u(s, t)$$

Different perturbations may not give same Floer complex, but these Floer complex will be homotopy invariant.

4.5 Fukaya categories

The objects of Fukaya category

$$\text{Ob}(\mathcal{F}) = L$$

where $\mu_L = 0$, with spin structures and $\pi_2(M, L) = 0$. And

$$\text{Hom}(L_i, L_j) = CF(L_i, L_j)$$

The Fukaya category does not have associativity but have A_∞-relations.

5 Floer homology and Fukaya categories II (2022-04-03 Lan Zhuoming)

5.1 Fukaya categories

We now continue to discuss the Fukaya categoires. We first define the product operation (composition) for homomorphisms. Let (M, w) be symplectic manifold, and (L_0, L_1, L_2) three Lagrangians which intersect transversally and $[w] \cdot \pi_2(M, L_i) = 0$, $\mu_{L_i} = 0$. We also assume that $2c_1(M) = 0$. We tries to define

$$\mu_2 : CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$$

To avoid some subtle discussion, we assume that we are working over characteristic 2. To define the above morphism, we only need to define for generators. For $p_2 \in \chi(L_1, L_2)$ and $p_1 \in \chi(L_0, L_1)$, then we define

$$p_2 \cdot p_1 = \sum_{q \in \chi(L_1, L_2), \text{ind}([u]) = 0} \#\mathcal{M}(p_1, p_2, q, [u], J)T^{w[u]}q$$

where $\mathcal{M}(p_1, p_2, q, [u], J)$ is the moduli space of pseudo-holomorphic maps with boundary on L_0, L_1, L_2 and boundary point p_1, p_2, q and homotopy class $[u]$.

Remark 5.1.1 *The dimension of moduli space* $\dim \mathcal{M}(p_1, \cdots, p_k, q, [u], J) = k - 2 + \text{ind}([u])$. *That is because first every points gives a freedom of choice but biholomoprhic map on disk makes the freedom minus 3 and* $\text{ind}([u])$ *brings new freedom of choice. Therefore in above case the sum is over* 0-*dimensional manifold and is meaningful.*

Theorem 5.1.2 $\partial(p_2 \cdot p_1) = \pm \partial p_2 \cdot p_1 + p_2 \cdot \partial p_1$

PROOF By Gromov compactness theorem, and vanishing of relative homotopy group, coefficient of q in $\partial(p_2 \cdot p_1) + \partial p_2 \cdot p_1 + p_2 \cdot \partial p_1$ equals to $\# \sum_{\text{ind}[u]=1} (\partial \mathcal{M}(p_1, p_2, r, [u], J))$ which equals 0.

We now define what is A_∞-categories.

Definition 5.1.3 *An* A_{infty}-***category*** *is following collection of data.*

1. *$Ob(A)$ the object of A.*

2. *For every $X_0, X_1 \in Ob(A)$, $\hom(X_0, X_1)$ graded vector space called morphisms between X_0, X_1.*

3. *A set of composition maps*

$$\mu_k : \hom(X_{k-1}, X_k) \otimes \cdots \otimes \hom(X_0, X_1) \to \hom(X_0, X_k)[2-k]$$

satisfying

$$\sum_{l=1}^{k} \sum_{j=0}^{k-l} (-1)^\dagger \mu_{k+1-l}(p_k, \cdots, p_{j+l+1}, \mu_l(p_{j+l, \cdots, p_{j+1}}), p_j, \cdots, p_1) = 0 \qquad (5.1)$$

where $\dagger = j + \deg(p_1) + \deg(p_j)$.

Definition 5.1.4 *Let* $\mathcal{A}, \mathcal{B}$ *tow* A_∞-*categories. An* A_∞-***functor*** $F : \mathcal{A} \to \mathcal{B}$ *is the datum of*

1. *A map $F : Ob(\mathcal{A}) \to Ob(\mathcal{B})$.*

2. *For all $n \geq 1$ and all $A_0, \cdots, A_n \in \mathcal{A}$ a graded amp*

$$F_n : \hom(A_{n-1,n}, A_n) \otimes \cdots \otimes \hom(A_0, A_1) \to \hom_\mathcal{B}(FA_0, FA_n)$$

of degree $1 - n$ satisfying similar relations of equation 5.1.

An A_∞-*functor is called an* A_∞-***equivalence*** *if the underlying cohomological level functor is an equivalence.*

Remark 5.1.5 *What we called equivalence are called quasi-equivalence in [Sei08].*

For $k = 1$, it gives $\mu_1^2 = 0$. Therefore μ_1 is a differential. And for $k = 2$, we have $\mu_1(\mu_2(p_2, p_1)) \pm \mu_2(\mu_1(p_2), p_1) - \mu_2(p_2, \mu_1(p_1)) = 0$. Here the plus-minus sign depends on the degree of p_1. $k = 2$ gives Lebniz rule.

To show that Fukaya category is an A_{infty}-category, we have to define the higher compostion maps. We define

$$\mu_k(p_k, \cdots, p_1) = \sum_{q \in \chi(L_0, L_k), \text{ind}([u])=2-k} \# \mathcal{M}(p_1, p_2, \cdots, p_k, q, [u], J) T^{w([u])} q$$

Here we have $\mathcal{M}(p_1, \cdots, p_k, q, [u], J)$ denote the moduli space of pseudo-holomorphic maps with boundary on $(L_0, \cdots, L_k)$ and boundary points $p_1, \cdots, p_k, q$ within homotopy class $[u]$. The A_∞-relation is deduced again from Gromov compactness theorem and no bubbling property.

Now we give the true definition of Fukaya categories.

Definition 5.1.6 *Let (M, w) be a symplectic manifold, with $2c_1(TM) = 0$. We define the* ***Fukaya category*** *associated to (M, w), denoted by $\mathcal{F}(M, w)$ by*

1. *$Ob(\mathcal{F})(M, w) = \{$compact closed Lagrangian manifolds with $\mu_L = 0\}$*

2. *$\hom(L_1, L_2) = CF(L, L', J_{L,L'}, H_{L,L'})$ where $H_{L,L'}, J_{L,L'}$ are Floer data to make Floer complexes well defined.*

3. *The compostion map μ_k are given as above, where (H,J) perturbation data is required to make the composition map well defined. In a neighborhood of $(L_0, \cdots, L_k)$, (H,J) and $H_{L,L'}, J_{L,L'}$ can be chosen to coincides.*

From definition, we can see that $\mathcal{F}(M,w)$ depends on some chosen data. But we can prove that

Proposition 5.1.7 *A Fukaya category is well-defined up to A_∞-equivalence.*

6 Homological mirror symmetry for elliptic curves I (2022-04-10 Zhang Nantao)

6.1 References

The main references are [Por15] and [PZ98].

6.2 Fukaya category of elliptic curves

We first state following facts.

1. In (complex) dimension 1 case, every closed 1-submanifold is Lagrangian.

2. The relative spin structure is a $\mathbb{Z}_2$ choice and can be suppressed.

3. Each Hamiltonian isotopy class of closed Lagrangian submanifold has a unique special Lagrangian representative. In elliptic curve case, it is a line with rational slope when lifted to universal cover $\mathbb{C}$.

Remark 6.2.1 *In general Calabi-Yau, uniqueness of special Lagrangian representative is true while the existence is unknown.*

Topologically, every elliptic curve is equivalent. The symplectic structure is determined by a real number. For mirror symmetry, we in fact need to determine a complex Kähler form $w_{\mathbb{C}} = \rho dx \wedge dy = (B + iA)dx \wedge dy$, such that $w = Adx \wedge dy$. The elliptic curve with complex Kähler form $\rho dx \wedge dy$. We always identify E^ρ with $\mathbb{C}/\langle 1, i \rangle$.

Our Fukaya category $Fuk(E^\rho)$ will be a little different from previous talk, but follows [PZ98].

1. Object: $Ob(Fuk(E^\rho))$ are graded, oriented, closed geodesics equipped with a flat complex vector bundle of monodromy with unit eigenvalue. That is a pair $\mathcal{L} = (L, \alpha, M)$, such that $z(t) = z_0 + te^{\pi i \alpha}$, orientation of L defined by $\beta \in (-\frac{1}{2}, \frac{1}{2}]$ and $\alpha - \beta \in \mathbb{Z}$.

2. Morphism:
$$\text{Hom}(\mathcal{L}, \mathcal{L}') := \begin{cases} \oplus_{p \in L \cap L'} \text{Hom}(M_p, M'_p) & \text{if } L \neq L' \\ \text{Hom}(M, M') & \text{if } L = L' \end{cases}$$

$\mathbb{Z}$ grading on morphism $\mathcal{L} \to \mathcal{L}'$ is given by Maslov index $\mu(\mathcal{L}, \mathcal{L}') = \lceil \alpha - \alpha' \rceil$.

We have a shifted functor $(L, \alpha, M)[1] = (L, \alpha + 1, M)$. Composition of morphisms are given by

$$m_k(u_1 \otimes \cdots \otimes u_k) = \sum_{p \in L_k \cap L_0} C(u_1, \cdots, u_k; p)p$$

where we assume that every $u_i = \sum_\mu t_{i\mu} p_{i\mu}$. And

$$C(u_1, \cdots, u_k; p) = \sum_{[\phi]} e^{2\pi i \int \phi^* w_{\mathbb{C}}} h_{\partial \phi}(u_1 \otimes \cdots \otimes u_k)$$

sum over $[\phi]$ in

$$C\mathcal{M}_{k+1}(X; L_0, \cdots, L_k; p_0, \cdots, p_k) := \{(\phi, S_k = (0 = \theta_0 < \theta_1 < \cdots < \theta_k < 2\pi)) \mid$$
$$\phi : D \to X \text{ is pseudo holomorphic}, \phi(e^{i\theta_j}) = p_j, \phi(e^{it}) \in L_j \quad \forall t \in (\theta_{j-1}, \theta_j)\}/\sim$$

where $(\phi, S_{k+1}) \sim (\phi', S'_{k+1})$ if there exists biholomorphic map $f : D \to D$ such that $\phi' \circ f = \phi$ and $f(e^{i\theta_j} = e^{i\theta'_j})$. And

$$h_{\partial\phi}(u_1 \otimes \cdots \otimes u_k) := P_{\gamma_k} \circ u_k \circ \cdots \circ u_1 \circ P_{\gamma_0}$$

where P_{γ_j} is parallel transport and $t \in [\theta_{j-1}, \theta_j]$. For simplicity, we may assuem $P_\gamma = \exp(Ml)$ where M is monodromy and l normalized length. We now consider $m_1 : \text{Hom}(\mathcal{L}_0, \mathcal{L}_1) \to \text{Hom}(\mathcal{L}_0, \mathcal{L}_1)$ by some pictorial argument, we have $C\mathcal{M}_2 = 0$ and therefore $m_1(u_1) = 0$ for every u_1. Therefore, by $\pm m_2(m_2(-,-),-) \pm m_2(-, m_2(-,-)) \pm m_1(m_3(-,-,-)) \pm m_3(m_1(-),-,-) \pm m_3(-, m_1(-),-) \pm m_3(-,-,m_1(-)) = 0$, we have $m_2(-, m_2(-,-)) = m_2(m_2(-,-),-)$, m_2 is associative. The simplest example, is $\rho = iA$, then $\mathcal{L}_d = (L_d, \alpha, M)$ where L_d had slope d passing through 0 and $\alpha \in (-\frac{1}{2}, \frac{1}{2}]$ and M trivial line bundle on L.

Then $L_0 \cap L_1 = \{e_1\}$, $L_1 \cap L_2 = \{e_1\}$ and $L_0 \cap L_2 = \{e_1, \cdots, e_2\}$. We have $m_2(e_1, e_1) = C(e_1, e_1, e_1)e_1 + C(e_1, e_1, e_2)e_2$, where

$$C(e_1, e_1, e_1) = \sum_{n \in \mathbb{Z}} e^{-2\pi An^2}$$

$$C(e_1, e_1, e_2) = \sum_{n \in \mathbb{Z}} e^{-2\pi A(n+\frac{1}{2})^2}$$

6.3 Derived category of coherent sheaves for elliptic curves

Now we define $E_\tau = \mathbb{C}/\langle 1, \tau \rangle$, or $E_q = \mathbb{C}^*/\mathbb{Z}$ with $\mathbb{Z}$ action generated by $z \to qz$, $q = e^{2\pi i\tau}$. To understand the derived category of coherent sheaves on E_q, we first study the line bundles on E_q. By covering $\mathbb{C}^* \to E_q$ every line bundle on E_q lifts to a line bundle on $\mathbb{C}^*$. But topologically every complex line bundle on $\mathbb{C}^*$ is trivial. So any holomorphic function $\phi : \mathbb{C}^* \to \mathbb{C}^*$ give rise to a line bundle

$$\mathscr{L}(q)(\phi) := \mathbb{C}^* \times \mathbb{C}/(u, v) \sim (uq, \phi(u)v)$$

Later I fixed q and omit it in subscript. All line bundles on E_q can be described in this way, and $\mathscr{L}(\phi) = \mathscr{L}(\phi')$ if and only if there exists some holomorphic $B : \mathbb{C}^* \to GL(\mathbb{C})$ such that $\phi'(u) = B(qu)\phi(u)B^{-1}(u)$. We have that

$$\mathscr{L}(\phi_1) \otimes \mathscr{L}(\phi_2) = \mathscr{L}(\phi_1\phi_2)$$

By directly check the transition function.

Let $Pic^d(C)$ denote the space of degree d line bundles on a curve C of genus g. Let $\mathscr{L}_d \in Pic^d(C)$, we have isomorphism

$$J(C) := Pic^0(C) \xrightarrow{\sim} Pic^d(C)$$

$$\mathscr{L} \to \mathscr{L} \otimes_{\mathscr{O}_C} \mathscr{L}_d$$

By Abel-Jacobi theorem, Jacobian variety of C is isomorphic to a complex torus of dimension g. In elliptic curve case, $J(E_\tau) = E_\tau$. Also, give $x \in E_\tau$, we have $\mathscr{L}_x$ is degree 1 line bundle on C. By above notation, we fix an degree 1 line bundle $\mathscr{L}$. Then every degree n line bundle is determined by $(t_x^*\mathscr{L}) \otimes \mathscr{L}^{n-1}$ whre t_x is the automorphism of E_{tau} given by translation. If $\mathscr{L} = \mathscr{L}(y)$, we have $t_x^*\mathscr{L} = \mathscr{L}(x+y)$.

We define $\mathscr{L} := \mathscr{L}(\phi_0)$ where $\phi_0(u) = q^{-1/2}u^{-1}$. Later we will show that $\mathscr{L}$ has a holomorphic section with a simple zero. So $\mathscr{L}$ has degree 1. Given two holomorphic line bundles $\mathscr{L}_1 = (t_x^*\mathscr{L}) \otimes \mathscr{L}^{n-1}$ and $\mathscr{L}_2 = (t_y^*\mathscr{L}) \otimes \mathscr{L}^{m-1}$. We need to determine homomoprhism between them.

$$\text{Hom}(\mathscr{L}_1, \mathscr{L}_2) = H^0(\mathscr{L}_1^* \otimes \mathscr{L}_2) = H^0((t_x^*\mathscr{L}^* \otimes t_y^*\mathscr{L}^* \otimes \mathscr{L}^{m-n}))$$

By using $t_\delta^*\phi_0^k = e^{-2\pi i\delta k}\phi_0^k$. We have

$$\mathscr{L}' = \mathscr{L}(t_x^*\phi_0^{-1} \cdot t_y^*\phi_0 \cdot \phi_0^{m-n}) = t_\delta^*(\mathscr{L}^k)$$

for $k = m - n$ and $\delta := \frac{y-x}{k}$ and where last equality holds for $k \neq 0$. For $k < 0$, we have $H^0(\mathscr{L}') = 0$, and for $k = 0$, we have $H^0(\mathscr{L}') = 0$ if $x \neq y$ and k if $x = y$.

We now determine $H^0(\mathscr{L}')$. This is done by theory of θ-functions to solve these questions. Consider the classical Jacobi **theta function**, $\theta_\tau : \mathbb{C} \to \mathbb{C}$ given by

$$\theta_\tau(z) := \sum_{m \in \mathbb{Z}} e^{\pi i[m^2\tau + 2mz]} = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}m^2}u^m$$

where $q = e^{2\pi i\tau}$, $u = e^{2\pi iz}$. Since $\Im(\tau) > 0$, the summation converges.

Proposition 6.3.1   *1. $\theta_\tau(z+1) = \theta_\tau(z)$*

2. $\theta_\tau(-z) = \theta_\tau(z)$

3. $\theta_\tau(z + \tau) = e^{-2\pi i[\tau + 2z]}\theta_\tau(z) = q^{-1/2}u^{-1}\theta_q(u)$

4. $\theta_\tau(\frac{\tau}{2} - z) = e^{2\pi iz}\theta_\tau(\frac{\tau}{2} + z)$

5. $\theta_\tau(x) = 0$ if and only if $z = \frac{1}{2} + \frac{\tau}{2} + (k + l\tau)$, $k, l \in \mathbb{Z}$ and these zeros are simple.

Recall $\phi_0 = q^{-1/2}u^{-1}$, so $\theta_\tau(z)$ gives a global section of $\mathscr{L}$.

PROOF 1,2,3,4 are easy from definition. Also,

$$\theta_\tau\left(\frac{1}{2} + \frac{\tau}{2}\right) = \sum_{m \in \mathbb{Z}}(-1)^m e^{\pi i(m^2 + m)\tau} = 0$$

To show it is a simple pole, we consider the integral of θ'/θ around the fundamental parallelogram using property 1 and 2. Such that

$$\frac{1}{2\pi i}\int_{parallelogram}\frac{d}{dz}(\log f)dz = 1$$

Let $\theta_\tau^{(x)}$ denote the θ function with zero at x, i.e.,

$$\theta_\tau^{(x)}(z) := \theta(z - \frac{1}{2} - \frac{\tau}{2} - x)$$

Theorem 6.3.2 *A function $R : E_\tau \to \mathbb{C}$ is meromorphic if and only if it can be written in the form*

$$R(z) = \frac{\prod_{i=1}^d \theta_\tau^{(x_i)}(z)}{\prod_{i=1}^d \theta_\tau^{(y_i)}(z)}$$

For $\{x_i\}, \{y_i\}$ finite set of $d > 0$ such that $\sum x_i - \sum y_i \in \mathbb{Z}$.

Now, we goes to compute $H^0(\mathscr{L}')$. We have following generalization of theta functions.

$$\theta[a, z_0](\tau, z) = \sum_{m \in \mathbb{Z}} e^{[(m+a)^2\tau + 2(m+a)(z+z_0)]}$$

Proposition 6.3.3   *1. $\theta[a + b, 0](\tau, z) = e^{\pi i[a^2\tau + 2az]}\theta[b, \tau z](\tau, z + a\tau)$*

2. Let $a, b \in \mathbb{Q}$, $n_1, n_2 \in \mathbb{Z}_{\geq 1}$, $k = n_1 + n_2$ and $c_j = jn_1 + a + b$. Then

$$\theta[a/n, 0](n_1\tau, z_1) \cdot \theta[b/n_2, 0](n_2\tau, z_2) = \sum_{j \in \mathbb{Z}/k\mathbb{Z}} \theta[\frac{c_j}{k}, 0](k\tau, z_1 + z_2)\theta[\frac{n_2c_j - kb}{n_1n_2k}, 0](n_1n_2k\tau, n_2z_1 - n_1z_2)$$

By setting $n_1 = n_2 = 1$, $a = b = 0$, $z = z_1 = z_2 - x$, we have

$$\theta_\tau(z)\theta_\tau(z + x) = \theta_{2\tau}(x)\theta_{2\tau}(2z + x) + \theta[\frac{1}{2}, 0](2\tau, x)\theta[\frac{1}{2}, 0](2r, 2z + x)$$

For $k = m - n > 0$ and $\delta := \frac{y-x}{k}$ we have

$$\text{Hom}(\mathscr{L}_1, \mathscr{L}_2) = H^0(t_\delta^* \mathscr{L}^k) = \text{span}_{\mathbb{C}}\{\theta[j/k, k\delta](k\tau, kz)\}_{j \in \mathbb{Z}/k\mathbb{Z}}$$

To see this, we only need to cconsider $H^0(\mathscr{L}^n)$, and we notice the dimension of $H^0(\mathscr{L}')$, are equal to $\deg(\mathscr{L}')$ by Riemann-Roch theorem. And theta functions are linear independent.

6.4 First example of HMS

We assume $\tau = iA$ and consider $\mathscr{L}_0 = \mathscr{O}$, $\mathscr{L}_1 = \mathscr{L}$ and $\mathscr{L}_2 = \mathscr{L}^2$ in $D^b(E_\tau)$ which are mapped by mirror functor to $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ discussed before. We have $\mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_1) = \mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2) = \mathrm{span}\{\theta_\tau(z)\}$. And

$$\mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_2) \cong \mathrm{span}\{\theta[0,0](2r, 2\tau), \theta[1/2, 0](2\tau, 2z)\}$$

We map $e_1 \in \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) \to \theta_\tau(z) \in \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_1)$, $e_1 \in \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \to \theta_\tau(z) \in \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_1)$, Also, $\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_2) \to \mathrm{Hom}(\mathscr{L}_0, \mathscr{L}_2)$ mapping $e_1 \to \theta[0,0](2r, 2z)$, $e_2 \to \theta[1/2, 0](2r, 2z)$. Recall, $m_2(e_1, e_1) = C(e_1, e_1, e_1)e_1 + C(e_1, e_1, e_2)e_2$ where

$$C(e_1, e_1, e_1) = \theta_{2\tau}(0)$$
$$C(e_1, e_1, e_2) = \theta[1/2, 0](2\tau, 0)$$

But we have

$$(\theta_\tau(z))^2 = \theta_{2\tau}(0)\theta_{2\tau}(2z) + \theta[1/2, 0](2\tau, 0)\theta[1/2, 0](2r, 2z)$$

Therefore the composition law holds for the mirror functor.

7 Homological mirror symmetry for elliptic curves II (2022-04-17 Zhang Nantao)

7.1 Derived category of coherent sheaves (continued)

We first introduce following theorems.

Theorem 7.1.1 *All indecomposable coherent sheaves over a Riemann surface are either an indecomposable vector bundle or a torsion sheaf supported at one point.*

Theorem 7.1.2 *Let C be a compact Riemann surface, finite direct sums of objects $\mathscr{F}[n]$, where $\mathscr{F}$ is an indecomposabl coherent sheaf on C, form a full subcategory of $D^b(C)$ which is equivalent to $D^b(C)$.*

The above theorem is proved using following lemma.

Lemma 7.1.3 *If $\mathcal{A}$ is an abelian category of homological dimension ≤ 1, then we have*

$$A^* \cong \oplus H^k(A^*)[k]$$

in $D^b(\mathcal{A})$.

By Serre duality, we have

$$\mathrm{Hom}(\mathscr{F}, \mathscr{G}) \cong \mathrm{Ext}^1(\mathscr{G}, \mathscr{F})^*$$

for coherent sheaves $\mathscr{F}, \mathscr{G}$.

We first consider vector bundles on E_τ. First, we have holomorphic vector bundles on E_τ given by $F_q(V, A) = \mathbb{C}^* \times V/(u, v) \sim (uq, Av)$ for $A \in GL(V)$. It is unique up to $B : \mathbb{C}^* \to GL(V)$, $A(u) = B(qu)A'(u)B(u)^{-1}$.

If $A = \exp(N)$, with N nilpotent with one dimensional kernel, then $F_q(V, \exp(N))$ is indecomposable. Let $\pi_r : E_{q^r} \to E_q$ r-fold covering. We have following theorem from Atiyah.

Theorem 7.1.4 *Every indecomposable holomorphic vector bundle on E_τ is of the form*

$$\pi_{r*}(\mathscr{L}_{q^r}(\phi) \otimes F_{q^r}(V, \exp(N)))$$

for some ϕ, N and r.

Let us ignore π_r first and consider $\mathscr{L}(\phi) \otimes F(V, \exp(N))$.

Lemma 7.1.5 *Let $\phi = (t_x^* \phi_0)\phi_0^{n-1}$ for some $n > 0$. Then for any nilpotent $N \in \mathrm{End}(V)$ there is a canonical isomorphism.*

$$\Psi_{\phi, N} : H^0(\mathscr{L}(\phi)) \otimes V \xrightarrow{\sim} H^0(\mathscr{L}(\phi) \otimes F(V, \exp(N)))$$
$$f \otimes v \to \exp(DN/n)f \cdot v$$

where $D = -u\frac{d}{du}$.

PROOF Let f be a section of $\mathscr{L}(\phi)$, then we have $f(qu) = \phi f(u)$ in the lifting. For w section of $L \otimes F(V, N)$ is equivalent to a holomorphic vector function $w(qu) = \phi(u)\exp(N)w(u)$. D commutes with $u \to uq$, i.e., $(Df)(uq) = D(f(uq))$. As f is a homogeneous equation of degree $-n$, we have $D(\phi f) = \phi(D + n)f$. As a result,

$$\exp(DN/n)f(qu)v = \sum \frac{1}{k!n^k}D^k(\phi f)N^k v = \phi \exp(N)(\exp\frac{DN}{n}f \cdot v)$$

Given two vector bundle $\mathscr{V}_i = \mathscr{L}(\phi_i) \otimes F(V_i, \exp(N_i))$ with $\phi_i = (t^*_{x_i}\phi_0)\phi_0^{n_i-1}$. Then we have

$$\mathrm{Hom}(\mathscr{V}_1, \mathscr{V}_2) = H^0(\mathscr{L}(\phi_1^{-1}\phi_2)) \otimes F(V_1^* \otimes V_2, \exp(\begin{pmatrix} N_1^* & 0 \\ 0 & N_2 \end{pmatrix})) = H^0(\mathscr{L}(\phi_1^{-1}\phi_2)) \otimes \mathrm{Hom}(V_1, V_2)$$

7.2 Fukaya-Kontsevich category

To establish the equivalence, we need to modify the Fukaya category to an additive category. We define $Fuk^0 := H^0(Fuk(E^\tau))$. That is

$$Ob(Fuk^0) = Ob(Fuk(E^\tau))$$

And for $L \neq L'$, we have

$$\mathrm{Hom}_{Fuk^0}(\mathcal{L}, \mathcal{L}') = \begin{cases} 0 & \text{if } \alpha' - \alpha \notin [0, 1) \\ \oplus_{p \in L \cap L'} \mathrm{Hom}(M_p, M'_p) & \text{if } \alpha' - \alpha \in [0, 1) \end{cases}$$

If $L = L'$, we have

$$\mathrm{Hom}_{Fuk^0}(\mathcal{L}, \mathcal{L}') = \begin{cases} 0 & \text{if } \alpha' - \alpha \neq 0, 1 \\ H^0(L, \mathrm{Hom}(M, M')) & \text{if } \alpha' = \alpha \\ H^1(L, \mathrm{Hom}(M, M')) & \text{if } \alpha' = \alpha + 1 \end{cases}$$

that is we only consider the degree 0 term in the hom space. Then we have

Lemma 7.2.1 *For $\mathcal{L}_1, \mathcal{L}_2$ in $Fuk^0(E^\tau)$, there exists a canonical isomorphism*

$$\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2[1]) \cong \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1)^*$$

Fuk^0 is still not additive, because for $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, then $\mathcal{L} \to \mathcal{L}_i$ and $\mathcal{L}_i \to \mathcal{L}$ can not be nontrivial at the same time. So we cannot have $\mathcal{L}$ in Fuk^0. So we have to do some formal completions. We define $\underline{\mathcal{A}}$ to be the category whose objects are ordered k-tuple of objects in $\mathcal{A}$ and whose morphisms are matrices of morphisms in $\mathcal{A}$. Composition is given by matrix multiplication and the zero tuple is the zero object. The **Fukaya-Kontsevich category** is defined to bey $FK(E^\tau) = \underline{Fuk^0(E^\tau)}$.

7.3 Proof of homological mirror symmetry

We want to prove the following theorem.

Theorem 7.3.1 *There exists a functor $\Phi_\tau : D^b(E_\tau) \to FK(E^\tau)$ is an equivalence of additive categories compatible with the shift functor.*

Remark 7.3.2 *Though we have shift functor, we don't have distinguished triangles in $FK(E^\tau)$, so Φ_r is not an equivalence of triangulated categories.*

The proof consists of three parts. First, we reduce to the case of holomorphic vector bundles $F(V, \exp(N))$ and $S = S(-a\tau - b, V, N)$, where $S(\zeta_0, V, N) := \mathscr{O}_{r,\zeta_0} \otimes V/\langle \zeta - \zeta_0 - \frac{1}{2\pi i}N \rangle$ where $\mathscr{O}_{r,\zeta_0}$ is deinfed by the exact sequence

$$\mathscr{O} \xrightarrow{(\zeta - \zeta_0)^r} \mathscr{O} \to \mathscr{O}_{r\zeta_0}$$

where $N^r = 0$ and $N^{r-1} \neq 0$ for $r = \dim V$. To do this, we consider $\pi_r : E_{q^r} \to E_q$ and $p_r : E^\tau \to E^{r\tau}$ mapping $(x, y) \to (rx, y)$. Let us consider the restriction of p_r to L with slope n/m with $gcd(n, m) = 1$. The degree of $p_r|_L$ is $d = gcd(n, r)$. For L of slope n/rm in $E^{r\tau}$, we have preimage $N = r/d$ connected components. We have pushforward

$$p_{r*} : FK(E^\tau) \to FK(E^{r\tau})$$

given by

$$p_{r*}(L, \alpha, M) \to (p_r(L), \alpha', p_{r*}M)$$

with α, α' both contained in $[k - \frac{1}{2}, k + \frac{1}{2})$ for some k. $p_{r*}M$ has monodromy

$$(v_1, \cdots, v_d) \to (v_2, \cdots, v_d, Mv_1)$$

$$
\begin{array}{ccc}
\operatorname{Hom}(\mathcal{L}, \mathcal{L}') & \xrightarrow{\ \cong\ } & \oplus_{\bar{x} \subset L \cap L'} \operatorname{Hom}(M_{\bar{x}}, M'_{\bar{x}}) \\
\downarrow{\scriptstyle p_{r*}} & & \downarrow \\
\operatorname{Hom}(p_{r*}\mathcal{L}, p_{r*}\mathcal{L}') & \xrightarrow{\ \cong\ } & Z
\end{array}
$$

And $Z := \oplus x \in p_r(L) \cap p_r(L') \operatorname{Hom}(\oplus_{\tilde{x} \in (p_r|_L)^{-1}(x)} M_{\tilde{x}}, \oplus_{\tilde{x}'(p_r|_{L'})^{-1}(x)} M'_{\tilde{x}'})$.

Also, we have pullback, $p_r^* : FK(E^{r\tau}) \to FK(E^{\tau})$ given by

$$p_r^*(L, \alpha, M) = \oplus_{k=1}^N (L^{(k)}, \alpha', (p_r^{(k)})^* M)$$

where $p_r^{(k)} := p_r|_{L^{(k)}}$ and $(p_r^{(k)})^* M$ is the local system pullback.

Proposition 7.3.3 *Let $p = t'_{n/m} \circ p_r$ for $n/m \in \mathbb{Q}$ and $t'_\delta(x, y) := (x - \delta, y)$ then*

$$\operatorname{Hom}(p^*\mathcal{L}, \mathcal{L}') \cong \operatorname{Hom}(\mathcal{L}, p^*\mathcal{L}')$$

and

$$\operatorname{Hom}(p_*\mathcal{L}, \mathcal{L}') \cong \operatorname{Hom}(\mathcal{L}, p^*\mathcal{L}')$$

the same is true for derived category of coherent sheaves when $\pi = t_{n\tau/m} \circ \pi_r$.

That is if we establish the mirror functor $\Phi : D^b(E_\tau) \to FK(E^\tau)$ on $L(\phi) \otimes F(V, \exp(N))$, we can extend to all indecomposable vector bundles by assigning

$$\Phi(\pi_{r*}A) := p_{r*}(\Phi(A))$$

And we also need to define

$$\operatorname{Hom}(\pi_{r_1*}\mathscr{E}_1, \pi_{r_2*}\mathscr{E}_2) \to \operatorname{Hom}(p_{r_1*}\Phi(\mathscr{E}_1), p_{r_2*}\Phi(\mathscr{E}_2))$$

To do this, we consider the fiber product $E_{r_1\tau} \times_{E_\tau} E_{r_2\tau}$ disjoint union of elliptic curves $E_{r\tau} \times \mathbb{Z}_d$, where $d = gcd(r_1, r_2)$ and $r = lcm(r_1, r_2)$. Let $\tilde{\pi}_{r_i,v} : E_{r\tau} \times \{v\} \to E_{r_i\tau}$ and $\tilde{p}_{r_i,v} : E_{r\tau} \times \{v\} \to E_{r_i\tau}$. Then we have $\tilde{p}_{r_2*} \circ \tilde{p}_{r_1}^* = p_{r_1}^* \circ p_{r_2*}$. Then we define

$$
\begin{array}{ccc}
\operatorname{Hom}(\pi_{r_1*}\mathscr{E}_1, \pi_{r_2*}\mathscr{E}_2) & \xrightarrow{\ \cong\ } & \oplus_{v=1}^d \operatorname{Hom}(\tilde{\pi}_{r_1,v}^*\mathscr{E}_1, \tilde{\pi}_{r_2,v}^*\mathscr{E}_2) \\
\downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \oplus\Phi} \\
\operatorname{Hom}(\Phi(\pi_{r_1*}\mathscr{E}_1), \Phi(\pi_{r_2*}\mathscr{E}_2)) & \xrightarrow{\ \cong\ } & \oplus_{v=1}^d \operatorname{Hom}(\Phi(\tilde{\pi}_{r_1,v}^*\mathscr{E}_1), \Phi(\tilde{\pi}_{r_2,v}^*\mathscr{E}_2)) \\
\downarrow{\scriptstyle \sim} & & \downarrow \\
\operatorname{Hom}(p_{r_1*}\Phi(\mathscr{E}_1), p_{r_2*}\Phi(\mathscr{E}_2)) & \longrightarrow & \oplus_{v=1}^d \operatorname{Hom}(\tilde{p}_{r_1,v}^*\mathscr{E}_1, \tilde{p}_{r_2,v}^*\mathscr{E}_2)
\end{array}
$$

By Serre duality on both side, the splitting of complex and Grothendieck vanishing, we only need to consider the the morphisms in $Coh(E_\tau) \subset D^b(E_\tau)$.

Second, we need to construct the morphism for $F(V, \exp(N))$ and $S = S(-a\tau - b, V, N)$. For $A = \mathscr{L}(\phi) \otimes F(V, \exp(N))$, $\phi = (t_{a\tau+b}^*\phi_0)\phi_0^{n-1}$. then $\Phi_\tau(A) = (L, \alpha, M)$ where

- L is the submanifold of E^τ with lift parametrized by $t \to (a + t, (n-1)a + nt)$.

- α is the unique α such that $-1/2 < \alpha < 1/2$.

- $M = e^{-2\pi i b} \exp(N)$.

For indecomposable torsion sheaf $S = S(-a\tau - b, V, N)$, we then $\Phi_r(S) = (L, \frac{1}{2}, e^{-2\pi i b}\exp(N))$ with L vertical with x-interception a. Now we consider the functor between morphisms. If $A_i = \mathscr{L}(\phi_i) \otimes F(V_i, \exp(N_i))$, $x_i = -a_i/n_i$, we define
$$\Phi : \text{Hom}(A_1, A_2) \to \text{Hom}_{FK(E^\tau)}(\Phi(A_1), \Phi(A_2))$$

Let $\theta = t^*_{\delta_{21}\tau + \beta_{21}} f_k^{(n_2-n_1)}$ for $\delta_{21} := \frac{n_2 x_2 - n_1 x_1}{n_2 - n_1}$ and $\beta_{21} := \frac{\beta_2 - \beta_1}{n_2 - n_1}$, $f_k^{(n_2-n_1)}(z) := \theta[\frac{k}{n_2-n_1}, 0]((n_2 - n_1)\tau, (n_2 - n_1)z)$. Then
$$\Phi(\theta \otimes f) = e^{\pi i \delta_{21}^2(n_1-n_2)}\exp(\delta_{21}(N_2 - N_1^* - 2\pi i(n_2 - n1)\beta_{21})) \cdot f \cdot e_k$$

where
$$e_k = (\frac{n_1 x_1 - n_2 x_2 + k}{n_1 - n_2}, \frac{n_1}{n_1 - n_2}[n_2(x_1 - x_2) + k])$$

for $k \in \mathbb{Z}/(n_1 - n_2)\mathbb{Z}$. If $n_2 = n_1$ then the morphism space is zero unless $\beta_2 = \beta_1$ and we have the map between morphism identity map.

For $S = S(-a\tau - b, V', N')$, then $\text{Hom}(S, A) = 0$ and $\text{Hom}(\Phi(S), \Phi(A)) = 0$ for $A = \mathscr{L}(t^*_{a\tau+\beta}\phi \cdot \phi_0^{n-1}) \otimes F(V, \exp(N))$ then $\text{Hom}(A, S) = \text{Hom}(V, V')$. The Lagraingian intersects at 1 point and therefore we have

$$\Phi_\tau : V' \otimes V^* \to V' \otimes V^*$$

$$\Phi_r = e^{-\pi i \tau(na^2 - 2a\alpha) + 2\pi i(a\beta + b\alpha - nab)}\exp[(na - \alpha)1_{V^*} \otimes N' + \alpha N^* \otimes 1_{V'}]$$

For $S_i = S(-a_i\tau - b_i, V_i, N_i)$, since they supported at a point, we only need to consider the case $a_1\tau + b_1 = a_2\tau + b_2$, then

$$\text{Hom}(S_1, S_2) = \{f \in \text{Hom}(V_1, V_2) \mid f \circ N_1 = N_2 \circ f\} = \text{Hom}(\Phi(S_1), \Phi(S_2))$$

By above construction, we explicitly describe the functor Φ for indecomposable object in $Coh(E_\tau)$.

Finally, we need to check the composition law holds, which we omitted.

7.4   Derived category of A_∞ categories

Now we discussion more on Fukaya categories. We want to construct a triangulated category from this A_∞-category. First, we construct the additive enhancement of A_∞-category $\mathcal{A}$, $\Sigma\mathcal{A}$.

$$Ob(\Sigma\mathcal{A}) = \{X = \oplus_{i \in I} V^i \otimes X^i\}$$

where V^i is graded vector space and I totally orderd. Morphisms are given by

$$\text{hom}_{\Sigma\mathcal{A}}(\oplus_{i \in I_0} V_0^i \otimes X_0^i, \oplus_{i \in I_1} V_1^j \otimes X_1^j) = \oplus_{i,j} \text{hom}(V_0^i, V_1^j) \otimes \text{hom}_{\mathcal{A}}(X_0^i, X_1^j)$$

and compositions are given by shifting compositions in $\mathcal{A}$.

Second, we consider the $tw\mathcal{A}$ as closure under extensions.

$$Ob(tw\mathcal{A}) = \{X, \delta_X\}$$

where $\delta_X \in \text{hom}^1_{\Sigma\mathcal{A}}(X, X)$, and $\delta_X = (\delta_{ij})$ a matrix for $\delta_{ij} \in \text{hom}^1_{\Sigma\mathcal{A}}(V^i \otimes X^i, V^j \otimes X^j)$. We require $\delta_{ij} = 0$ for all $i \geq j$, and

$$\sum_{t=1}^{\infty} \mu_{\Sigma\mathcal{A}}^r(\delta_X, \cdots, \delta_X) = 0$$

since δ is strictly upper triangle, the sum is finite. The morphisms are given by

$$\text{hom}((X, \delta), (X', \delta')) = \text{hom}_{\Sigma\mathcal{A}}(X, X')$$

The composition is given by

$$\mu_{tw\mathcal{A}}^d(a_d, \cdots, a_1) = \sum_{i_1, \cdots, i_d} \mu_{\Sigma\mathcal{A}}^{d+i_0+\cdots+i_d}(\delta_{X_d}, \cdots, \delta_{X_d}, a_d, \delta_{X_{d-1}}, \cdots, \delta_{X_{d-1}}, \cdots, a_1, \cdots, \delta_{X_0}, \cdots, \delta_{X_0})$$

there are i_d times δ_{X_d}.

The purpose of above enhancement is to make direct sum and shift $(S(-) = (-) \otimes k[1])$ possible. Further, we have mapping cone for $c \in \hom_{tw\mathcal{A}}(Y_0, Y_1)$

$$C = SY_0 \oplus Y, \delta_c = \begin{pmatrix} S(\delta_{Y_0}) & 0 \\ -S(c) & \delta Y_1 \end{pmatrix}$$

such that $D(\mathcal{A}) = H^0(tw\mathcal{A})$ is an triangulated category, called **derived category** of $\mathcal{A}$.

Furthermore, we want the category to be **split closed**, that is for $p \in \mathrm{Hom}_\mathcal{A}(Y, Y)$, $p^2 = p$ idempotent, then $Y = Z \oplus Z'$ respect to p. As derived category of coherent sheaves is split closed, we have to take the **split closure** of $H^0(tw\mathcal{A})$ by formally adding all splittings to get

$$D^\pi(\mathcal{A}) = PH^0(tw\mathcal{A})$$

called **split closed derived category**. P is the **Karoubi completion**. For $\mathcal{A}$ category, we have $Ob(P\mathcal{A}) = (Y, p)$, where p is idempotent morphism. $\mathrm{Hom}_{P\mathcal{A}}((Y_0, p_0), (Y_1, p_1)) = p_1 \mathrm{Hom}(Y_0, Y_1)p_0$.

In [Sei15; She15], they prove there exists an equivalence of triangulated categories

$$D^\pi(Fuk(M^n)) \cong \hat{\psi}^* D^b Coh(N^n_{nov})$$

where $\psi \in \mathbb{C}[\![r]\!]$, $\psi(0) = \pm 1$, and ψ^* is an automorphism of $\mathbb{C}[\![r]\!]$, such that $\psi^*(r) = r\psi(r)$. And $\hat{\psi}^*$ is a lifting of ψ^* to Novikov field.

$$\Lambda = \{\sum_{j=0}^\infty c_j r^{\lambda_j}, \lambda_j \in \mathbb{R}, \lim_{j \to \infty} \lambda_j = +\infty\}$$

$M^n \subset \mathbb{C}P^{n-1}$ is $(n-2)$-dimensional Calabi-Yau hypersurface. And $\tilde{N}^n_{nov} = \{w_{nov} = 0\} \subset \mathbb{P}^{n-1}_\Lambda$ where $w_{nov} = u_1 \cdots u_n + r \sum_{j=1}^n u_j^n \in \Lambda[u_1, \cdots, u_n]$, with $\Gamma_n := (\mathbb{Z}_n)^n/(1, \cdots, 1)$ acting on $\mathbb{P}^n_\Lambda$ by multiplication by root of unity. And $\Gamma_n^* = \ker(\Gamma_n \to \mathbb{Z}_n)$ acts on $\tilde{N}^*_{nov}$. Then we define

$$N^n_{nov} := \tilde{N}^n_{nov}/\Gamma_n^*$$

is the stacky root. So we have $D^b Coh(N^n_{nov}) \cong D^b(Coh^{\Gamma_n^*}(\tilde{N}^n_{nov}))$.

Remark 7.4.1 *It can be made an (quasi)equivalence of A_∞-categories by choosing dg enhancement of right hand side and making split closure of $twFuk(M^n)$ on the right hand side.*

Remark 7.4.2 *The equivalence is only proved for one side, that A-side of M^n is equivalent to B-side of N^n_{nov}. Homological mirror symmetry also conjectures the dual part that is the B-side of M^n is equivalent to the A-side of N^n_{nov}, which is proved for qurtic surface in [SS21] but not for higher dimensional case.*

8 Constructible sheaves and the Fukaya category (2022-04-24 Zhou Jiawei)

8.1 References

We discuss paper [NZ09].

8.2 Constructible sheaves

We always assume that X is a real analytic manifold. The main theorem is that

Theorem 8.2.1 $D_c(X) = D(Sh(X)) \xrightarrow{qis} DFuk(T^*X)$ *where $Sh(X)$ is the category of constructible sheaves.*

Definition 8.2.2 *A sheaf $\mathscr{F}$ is called **constructible** if there exists a Whitney stratification $X = \coprod X_i$ such that $\mathscr{F}|_{X_i}$ is a local system (and finitely generated).*

We consider the dg category $Sh_{naive}(X)$ where the objects are complex of sheaves with bounded constructible cohomology. And morphisms are internal hom between complexes. Let $\mathcal{N} \subset Sh$ subcategory of acyclic objects. And we define $Sh(X)$ as the dg quotient $Sh_{naive}(X)/\mathcal{N}$. The proof of the main theorem consists of three part, equivalence of $D(Sh(X))$ and $D(Open(X))$, equivalence of $Open(X)$ and $Mor(X)$ and equivalence of $Mor(X)$ and $Fuk(T^*X)$.

8.3 Category $Open(X)$

We first define $Open(X)$. The objects are pairs $\mathcal{U} = (U, m)$, where $U \subset X$ is open and $m : X \to \mathbb{R}^{\geq 0}$ is the defining function for $X \backslash U$, i.e. $m(x) = 0$ if and only if $x \in X \backslash U$. The morphisms between pairs $\mathcal{U}_0, \mathcal{U}_1$ are relative de Rham complex $\hom_{Open(X)}(\mathcal{U}_0, \mathcal{U}_1) = (\Omega^*(\bar{U}_0 \cap U_1, \partial U_0 \cap U_1, d))$. We consider the morphism $p : Open(X) \to Sh(X)$, $(U, m) \to i_* \mathbb{C}_U$, where $\mathbb{C}_U$ is the locally constant sheaf of complex value on U and $i : U \hookrightarrow X$ the inclusion.

Proposition 8.3.1 $D(Open(X)) = D(Sh(X))$.

Firstly, we have the following lemma.

Lemma 8.3.2 $\hom_{Sh(X)}(i_{0*}\mathbb{C}_{U_0}, i_{1*}\mathbb{C}_{U_1}) \overset{qis}{\cong} (\Omega^*(\bar{U}_0 \cap U_1, \partial U_0 \cap U_1, d))$.

Assuming the above lemma, the result follows from the following proposition.

Proposition 8.3.3 *Any objects of $Sh(X)$ is isomorphic to an object obtained by shifting of $i_* \mathbb{C}_U$, U open and $i : U \hookrightarrow X$ inclusion, by iteratively forming cones.*

PROOF We have $Z \subset X$, $i : U \hookrightarrow Z$ open and $j : Y = Z \backslash U \hookrightarrow Z$ closed. For all $\mathscr{F}$ sheaf on Z, there exists a distinguished triangle
$$j_* j^! \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to$$

Using above fact, we first claim that any object in $Sh(X)$ is isomorphic to an object shifting by standard object by iteratively forming cones. The standard objects are those of the form $i_* \mathscr{L}_Y$, whre $\mathscr{L}_Y$ is a local system on $Y \subset X$ and $i : Y \hookrightarrow X$. Let S strata with respect to which $\mathscr{F}$ is constructible. We define $S_k = \cup\{\text{strata of dimension } k\}$, and $S_{<k} = \cup\{\text{strata of dimension } \leq k\}$. We denote $i_k : S_k \to X$ and $j_k : S_{<k} \to X$ suppose $\dim X = n$, then we have distinguished triangle
$$j_{<n*} j^!_{<n} \mathscr{F} \to \mathscr{F} \to i_{k*} i^*_k \mathscr{F} \to$$

The last term is standard. Let's denote the first term by $\mathscr{F}_n$. Then we have
$$j_{<n-1*} j^*_{<n-1} \mathscr{F}_n \to \mathscr{F}_n \to i_{n-1*} i^*_{n-1} \mathscr{F}_n \to$$

By induction we prove our claim.

It remains to show when $\mathcal{T}$ is a stratification and T strata, then $i_* \mathbb{C}_T$ can be obtained by $i_* \mathbb{C}_U$ for U open. To prove this, we define $Star(T) = \cap\{\text{strata whose closure containing } T\}$ which is open. And we denote $Star'(T) = Star(T) \backslash T$ which is also open. $s : Star(T) \to X$ and $s' : Star'(X) \to X$, $j : T \to X$, then we have distinguished triangle
$$j_* j^! \mathbb{C}_{Star(T)} \to s_* \mathbb{C}_{Star(T)} \to s_* \mathbb{C}_{Star'(T)} \to$$

But we have $j^! \mathbb{C}_{Star(T)}$ is isomorphic to some shift of $\mathbb{C}_T$, and thus finish our proof.

8.4 Category $Mor(X)$

We now define the category $Mor(X)$ whose objects are the same as $Open(X)$ and the morphisms are generated by
$$\mathbb{C}\{Crit(X_{m_0 \geq \eta_0} \cap X_{m_1 > \eta_1}, f_1 - \epsilon_0 f_0)\}$$
where $f_i = \log m_i : U_i \to \mathbb{R}$ and $\eta_1, \epsilon_0, \eta_0$ is to gurantee finding a metric g such that $(f_1 - \epsilon_0 f_0, g)$ is direct pair on $X_{m_0 \geq \eta_0, m_1 \geq \eta_1}$. We call $(f_1 - \epsilon f_0, g)$ a **direct pair** if $\nabla(f_1 - \epsilon f_0)$ is inward pointing along $X_{m_1 = \eta_1}$ and outward pointing along $X_{m_0 = \eta_0}$.

To define the A_∞ structure on $Mor(X)$, we need to introduce **based metric ribbon tree**, (T, i, v_0, λ), where T finite tree with $d + 1$ vertices, with no vertices has exact 2 edges, $i : T \to D \subset \mathbb{R}^2$ where D is the closed disk and vertices are mapped to ∂D and edges are mapped to D°. v_0 is a chosen end vertex and $\lambda : \{internal edge\} \to \mathbb{R}^+$. Orientation of edges are given by the minimal path point to root vertex. Label the $d + 1$ components in $D \backslash T$ with number $\in \mathbb{Z}/(d+1)\mathbb{Z}$ counterclockwisely. Starting from 0, the left part of the edge terminate at v_0.

Suppose $i \in \mathbb{Z}/(d+1)\mathbb{Z}$, $U_i \subset X$ open, and $f_i : U_i \to \mathbb{R}$ extended to a neighbhorhood of $\bar{U}_i$ and critical point of $f_{i+1} - f_i$ is nondegenerate, and lies in $U_i \cap U_{i+1}$, such that g is a metric and $(f_{i+1} - f_i, g)$ is a direct pair. If such g exists, call $\{(U_i, f_i)\}$ **transverse collection**. A pair $(T, i, v_0, \lambda), \tau$ is a **gradient tree** where $\tau : T \to X$ is continuous such that

1. For all vertex $v \in T$, and e external edge containing v, we have

$$\tau(v) \in Crit(U_{r(e)} \cap U_{l(e)}, f_{r(e)} - f_{l(e)})$$

where $l(e)$ and $r(e)$ are left and right side of the edge e.

2. Identifying internal edge $e \cong [0, \lambda(e)]$, we have

$$\tau|_e = -\nabla(f_{l(e)} - f_{r(e)})$$

3. Identifying external edge $e \cong (-\infty, 0]$,

$$\tau|_e = -\nabla(f_{r(e)} - f_{l(e)})$$

Now fix $x_i \in Crit(U_i \cap U_j, f_{i+1} - f_i)$, we have the moduli space of gradient trees

$$\mathcal{M}(T; f_0, \cdots, f_d, x_0, \cdots, x_d)$$

where the vertices of T are mapped to x_i. We now define the A_∞ structure on $Mor(X)$ by

$$\mu^1(x_0) = \sum_T \sum_{x_1} \deg \mathcal{M}(T; \epsilon_0 f_0, f_1, x_0, x_1) x_1$$

where first summation over T with $1 + 1$ vertices. If the dimension $\mathcal{M}$ is nonzero, the degree is 0. For higher composition,

$$\mu^d(x_0, \cdots, x_{d-1}) = \sum_T \sum_{x_d} \deg \mathcal{M}(T; \epsilon f_0, \cdots, \epsilon_{d-1} f_{d-1}, f_d, x_0, \cdots, x_d) x_d$$

Proposition 8.4.1 *We have quasi-equivalence between A_∞ categories $Open(X) \to Mor(X)$.*

To show that $Open(X)$ is quasi-equivalent to $Mor(X)$, we have to use idea of Kontsevich-Soibelman by constructing map $i : Open(X) \to Mor(X)$, $\pi : Mor(X) \to Open(X)$, $h : Mor(X) \to Mor(X)$ such that $\pi i = \mathrm{id}_{Open(X)}$ and $i\pi = dh + hd$. In grading 0 term, it is basically the isomorphism between de Rham cohomology and Morse cohomology. The details are omitted here.

Finally, we have following proposition.

Proposition 8.4.2 *We have quasi-equivalence between A_∞-categories $Mor(X) \to Fuk(T^*X)$.*

The morphsim is given by mapping (U_i, m_i) to perturbation of $L_{U_i, f_i} := $ (zero section of $T^*X)|_{U_i} + \Gamma_{df_i}|_{U_i}$. The proof goes by comparing the moduli space.

All together, we have

$$D(Sh(X)) \to D(Open(X)) \to D(Mor(X)) \to DFuk(T^*X)$$

the equivalence and gives our main theorem.

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