INTERSECTION THEORY

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1 INTRODUCTION

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To motivate the study, we first give some examples or applications in intersection theory.

1.1 EXAMPLE. Bezout theorem.

For more visit https://zenith-john.github.io/post/algebraic_geometry_seminar_2022_fall.

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1.2 EXAMPLE. (27 lines on cubic surface) To prove this theorem, we need to consider Grassmannian $Gr(1,3)$ parametrizing lines in $\mathbb{C}P^1$. Then given any point $z \in Gr(1,3)$ we can write four linear equations on $\mathbb{A}^{20}$ parametrizing cubic surfaces, such that four equations are spontaneously zero if and only if this line is contained in the corresponding cubic surface. The equations can be glued to give a rank 4 vector bundles over $Gr(1,3)$. Every element in $\mathbb{A}^{20}$ gives a global section of $Gr(1,3)$. To get number 27, we need to compute the intersection number of a general section and zero section of the vector bundle.

1.3 EXAMPLE. (Gromov-Witten theory) It is a natural question to determine the number of genus g curves in a scheme X probably with some constraints. To do so, we first construct the moduli space $M_g(X)$ parametrizing all the genus g curves in X. Every constraint determines an algebraic cycles in the moduli space $M_g(X)$. To get final number, we just take the intersection of all algebraic cycles. If the dimension of intersections is zero, then we conclude the number of points is the number of the genus g curves with constraints.

In this introduction, we also want to discuss the relation or difference between algebraic geometry and algebraic topology. We may consider an embedded non-contractible $M \cong S^1$ in 2-dimensional real torus T^2. How do we compute self-interesetion of M? There are basicly two ways to do so: by homological methods and by perturbation. However, we do not have homology in algebraic geometry. On the other hand, it is also only possible to perturbate the submanifold to tranversal intersection only in differential manifolds. For example, we can consider

$$i : \mathbb{C}P^1 \to \mathcal{O}_{\mathbb{C}P^1}(-1) \cong Bl_0(\mathbb{A}^2)$$

Here $E = i(\mathbb{C}P^1)$ is the zero section and exceptional divisor. A standard result of complex geometry says the self-intersection $E \cdot E = -1$. However, E can not be deformed in the realm of algebraic geometry, because $\Gamma(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(-1)) = 0$. Therefore, we can not explain "intersection number" by perturbation.

Instead, much of the information is not contained in the $U \cap V$ part bu in the higher homological part. We have Serre intersection formula.

1.4 THEOREM. *Given a regular scheme* X *and subscheme* $Y, Z \subset X$ *with defining ideal* $\mathscr{I}, \mathscr{J}$. *Then intersection multiplicity at a generic point* x *of* $Y \cap Z$ *is*

$$m(x, Y, Z) = \sum_{i \geq 0} (-1)^i l_{\mathcal{O}_{X,x}} \operatorname{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathscr{I}_x, \mathcal{O}_{X,x}/\mathscr{J}_x)$$

Here l *denotes length.*

2 ALGEBRAIC CYCLES

2.1 *Cycles*

Let X be an algebraic scheme. A *k-cycle* on X is a finite formal sum

$$\sum n_i[\mathrm{V}_i]$$

where V_i are k-dimensional subvarieties (i.e. irreducible reduced scheme) of X and n_i are integers. The group of k-cycles, denoted by $\mathrm{Z}_k\mathrm{X}$, is a free abelian group. For any $(k+1)$-dimensional subvariety W of X, and any $r \in \mathrm{R}(\mathrm{W})^*$ a rational function, we can associate it with a k-cycle

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_{\mathrm{V}}(r)[\mathrm{V}]$$

The sum is over all codimensional 1 subvarieties V of W. The sum is always finite. $\mathrm{ord}_{\mathrm{V}}(r)$ denote the vanishing order of r at V. Let A be the local ring $\mathrm{A} := \mathscr{O}_{\mathrm{V,W}}$. If $r \in \mathrm{A}$, then $\mathrm{ord}_{\mathrm{V}}(r) = l_{\mathrm{A}}(\mathrm{A}/(r))$, and extend to R(X) by

$$\mathrm{ord}_{\mathrm{V}}(a/b) = \mathrm{ord}_{\mathrm{V}}(a) - \mathrm{ord}_{\mathrm{V}}(b)$$

A k-cycle α is *rationally equivalent to zero*, written as $\alpha \sim 0$, if there are a finite number of $(k+1)$-dimensional subvarieties W_i of X, and $r_i \in \mathrm{R}(\mathrm{W}_i)^*$, such that

$$\alpha = \sum [\mathrm{div}(r_i)]$$

The cycles rationally equivalent to zero form a subgroup $\mathrm{Rat}_k\,\mathrm{X}$ of $\mathrm{Z}_k\mathrm{X}$. The *group of k-cycles modulo rational equivalence* on X is the quotient group

$$\mathrm{A}_k\mathrm{X} = \mathrm{Z}_k\mathrm{X}/\,\mathrm{Rat}_k\,\mathrm{X}$$

and its element is called *cycle class*.

We denote

$$\mathrm{Z}_*\mathrm{X} = \oplus \mathrm{Z}_k\mathrm{X}, \quad \mathrm{A}_*\mathrm{X} = \oplus \mathrm{A}_k\mathrm{X}$$

A cycle is called *positive* if it is nonzero and all coefficients n_i are positive. A cycle class is called *positive* if it can be represented by a positive cycle.

2.1 EXAMPLE.

$$\mathrm{A}_k\mathrm{X} = \mathrm{A}_k\mathrm{X}_{red}$$

2.2 EXAMPLE. If X is nonsingular along V, then $\mathscr{O}_{\mathrm{V,X}}$ is a DVR. For $r \in \mathrm{R}(\mathrm{X})^*$, $r = ut^m$, $u \in \mathrm{A}^*$, $m \in \mathbb{Z}$ and t uniformizer. Then $\mathrm{ord}_{\mathrm{V}}(r) = m$. In general, we only have

$$\mathrm{ord}_{\mathrm{V}}(r) \geq \max\{n \mid r \in \mathscr{M}^n_{\mathrm{V,X}}\}$$

2.3 EXAMPLE. Let $\tilde{X} \to X$ be normalization of X. Then

$$\mathrm{ord}_V(r) = \sum \mathrm{ord}_{\tilde{V}}(r)[R(\tilde{V}) : R(V)]$$

where sum is taken over all subvarieties $\tilde{V}$ of $\tilde{X}$ which map onto V.

2.4 EXAMPLE. Let X_1 and X_2 be closed subschemes of X, then there are exact sequence

$$A_k(X_1 \cap X_2) \to A_k X \oplus A_k X_2 \to A_k(X_1 \cup X_2) \to 0$$

2.2 *Operations of algebraic cycles*

Let $f : X \to Y$ be a proper morphism. For subvariety $V \subset X$, the image $f(V)$ is closed subvariety W of Y. Then we define

$$\deg(V/W) = \begin{cases} [R(V) : R(W)] & \text{if } \dim W = \dim V \\ 0 & \text{if } \dim W < \dim V \end{cases}$$

Define $f_*[V] = \deg(V/W)[W]$ which extends to

$$f_* : Z_k X \to Z_k Y$$

and induces

$$f_* : A_k X \to A_k Y$$

(For proof, see [Ful98, Theorem 1.3, Proposition 1.4])

If X is proper over S = Spec K and $\alpha = \sum_P n_P[P]$ is a zero-cycle on X, then we define *degree* of α by

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P[R(P) : K]$$

We can define

$$\int_X : A_* X \to \mathbb{Z}$$

by defining $\int_X \alpha = 0$ for $\alpha \in A_k X, k > 0$.

Now let $f : X \to Y$ be a flat morphism (of relative dimension n).

2.5 NOTATION. In later part, when we say $f : X \to Y$ is flat, we assumes f is flat of relative dimension n for some $n \in \mathbb{Z}$.

We then define $f^*[V] = [f^{-1}(V)]$ of dimension $\dim V + n$. Then it extends to a homomorphism

$$f^* : Z_k Y \to Z_{k+n} X$$

and induces

$$f^* : A_k Y \to A_{k+n} X$$

(For proof, see [Ful98, Theorem 1.7])

We can also define the exterior product

$$Z_k X \otimes Z_k Y \to Z_{k+l}(X \times Y)$$

given by

$$[V] \times [W] \to [V \times X]$$

and induces

$$A_k X \otimes A_l Y \to A_{k+l}(X \times Y)$$

2.6 EXAMPLE. (Properness is essential) We may consider $X = \mathbb{P}^1_K \cup_{\mathbb{A}^1_K} \mathbb{P}^1_K \to \operatorname{Spec} K$, and rational function $r = x/y$ on X. Then we have $f_*[\operatorname{div}(r)] \neq 0$.

2.7 EXAMPLE. Let X be a nonsingular curve of genus g. Then $A_0 X = \operatorname{Pic}(X)$. Notice that for $g > 0$, $A_0 X$ is not finite generated, in contrast with the case of homology.

2.8 EXAMPLE. Let $f : X' \to X$ be a finite and flat morphism; each point of X has an neighborhood U such that coordinate ring of $f^{-1}(U)$ is a finite generated free module over coordinate ring of U. We say f has degree d if the rank of this module is d for all such U. Then

$$f_* f^* [V] = d[V]$$

2.9 EXAMPLE. (Proposition 1.8 of [Ful98]) Let Y be a closed subscheme of X and $U = X - Y$. Let $i : Y \to X$, $j : U \to X$ be the inclusion. Then the sequence

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j^*} A_k U \to 0$$

is exact.

Proof. Since any subvariety V of U extends to a subvariety $\bar{V}$ of X, so the sequence

$$Z_k Y \to Z_k X \to Z_k U \to 0$$

is exact. To see the middle exactness passes to cycle classes, we show that if $\alpha \in Z_k X$ such that $j^* \alpha \sim 0$, then it is in the image of i_*. By assumption, we have

$$j^*\alpha = \sum[\mathrm{div}(r_i)]$$

for $r_i \in R(W_i)^*$. Since $R(\bar{W}_i) = R(W_i)$, we have $\bar{r}_i \in R(\bar{W}_i)$ and

$$j^*(\alpha - \sum[\mathrm{div}(\bar{r}_i)]) = 0$$

in $Z_k U$. Therefore

$$\alpha - \sum[\mathrm{div}(\bar{r}_i)] = i_*\beta$$

for some $\beta \in Z_k Y$, which complete the proof. □

2.10 EXAMPLE. (Propositon 1.7 of [Ful98]) Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

be a fibre square, with g flat and f proper. Then

$$f'_* g'^* \alpha = g^* f_* \alpha$$

2.11 EXAMPLE. (Proposition 1.9 of [Ful98]) Let $p : E \to X$ be an affine bundle of rank n. Then the flat pull-back

$$p^* : A_k X \to A_{k+n} E$$

is surjective for all k.

Sketch of proof. Continuously applying exact sequence

$$A_k Y \to A_k X \to A_k U \to 0$$

and Noetherian reduction, we may assume that X is affine. By consider composition, $X \times \mathbb{A}^n \to X \times \mathbb{A}^{n-1} \to X$, we may assume $E = X \times \mathbb{A}^1$. Then just do the commutative algebra stuff to figure it out. □

2.12 EXAMPLE. $A_k(\mathbb{A}^n) = \begin{cases} 0 & \text{for } k < n \\ \mathbb{Z} & \text{for } k = n \end{cases}$

2.13 EXAMPLE. $A_k(\mathbb{P}^n) = \mathbb{Z}$ for $k \le n$.

2.14 EXAMPLE. Let $H \subset \mathbb{P}^n$ be a hypersurface of degree d. Then

$$A_{n-1}(\mathbb{P}^n - H) = \mathbb{Z}/d\mathbb{Z}$$

2.15 EXAMPLE.

$$(f \times g)_*(\alpha \times \beta) = f_*\alpha \times f_*\beta$$

$$(f \times g)^*(\alpha \times \beta) = f^*\alpha \times f^*\beta$$

2.3 *Alternative description of rational equivalence*

Let X be any scheme and $X_1, \cdots, X_t$ irreducible components of X. The local rings $\mathcal{O}_{X_i,X}$ are zero dimensional. The *geometric multiplicity* m_i is defined to be

$$m_i = l_{\mathcal{O}_{X_i,X}}(\mathcal{O}_{X_i,X})$$

The *fundamental cycle* [X] of X is the cycle

$$[X] = \sum_{i=1}^{t} m_i[X_{i,red}]$$

Now we give another description of rational equivalence.

2.16 PROPOSITION. *(Proposition 1.6 of [Ful98]) A cycle α in Z_kX is rationally equivalent to zero if and only if there are $(k+1)$-dimensional subvarieties $V_1, \cdots, V_t \subset X \times \mathbb{P}^1$ such that the projection $V_i \to \mathbb{P}^1$ is dominant, with*

$$\alpha = \sum_{i=1}^{t} [V_i(0)] - [V_i(\infty)]$$

in Z_kX.

3 INTERSECTION WITH DIVISORS

3.1 *Divisors*

Lecture 2 20th September 2022 By Liyuan Ye

A *Weil divisor* on a variety X is a codimension 1 cycle, i.e. finite formal sum of codimension 1 subvarieties, e.g. $[D] = \sum n_V[V]$ is a codimensional 1 subvariety. The Weil divisor form a group $Z_{n-1}X$ and $A_{n-1}X$.

A *Cartier divisor* is a data $\{(U_\alpha, f_\alpha)\}$ such that

1. U_α form an open covering.
2. f_α is local equation in $\Gamma(U_\alpha, \mathcal{K}^*)$.
3. f_α/f_β is unit in $\Gamma(U_\alpha, \mathcal{O}^*)$.

When X is a variety, V a subvariety. Then $\mathrm{ord}_V f_\alpha$ is well-defined and

$\mathrm{ord}_V f_\alpha = \mathrm{ord}_V f_\beta$ if $U_\alpha \cap U_\beta \neq \emptyset$. This gives Hom:

$$C \to \sum_V \mathrm{ord}_V(f_\alpha)[V]$$

and the sum is right is finite.

Any $f \in R(X)^*$ determines a principal Cartier divisor $\{(U_\alpha, f|_{U_\alpha})\}$. Two Cartier divisors are *linear equivalent* if and only if they differ a principal Cartier divisor. Denote Pic(X) as group of Cartier divosr modulo principal divisor. And we have a map

$$\mathrm{Hom} : \mathrm{Pic}\, X \to A_{n-1} X$$

3.1 REMARK. The Hom is in general neither injective nor surjective.

The *support* of Cartier divisor D denote $|D|$, supp(D) is the union of subvariety Z of X such that local equation D in the local ring $\mathscr{O}_{Z,X}$ is not a unit. The support of D is closed subset of X.

A *pseudo-divisor* on a scheme X is a triple (L, Z, s) such that

1. L is a line bundle on X.
2. Z is a support.
3. s a section of L and no where vanishing on X/Z.

3.2 DEFINITION. (L', Z', s') and (L, Z, s) define the same pseudo divisor if $Z = Z'$, and $\sigma|_{X-Z} : L \to L'$ is an isomorphism and $\sigma|_{X-Z}(s) = s'$.

If $Z = X$, then we call (L', X', s') and (L, Z, s) equivalent of $L' \cong L$.

A Cartier divisor D gives a pseudo-divisor $(\mathscr{O}_X(D), |D|, s_D)$. And we say that a Cartier divisor D *represents* a pseudo-divisor (L, Z, s) if $|D| \subset Z$, and exists an isomorphism $\mathscr{O}_X(D) \to L$ takes $s_D \to s$.

3.3 REMARK. Here we allow Z to be larger than $|D|$. And if $Z = X$, then all linear equivalent Cartier divisor represents the same pseudo-divisor.

3.4 LEMMA. *If* X *is a variety, then any pseodo-divisor* (L, Z, s) *can be represented by some Cartier divisor. Moreover,*

1. *If* $Z \neq X$, *then* D *is uniquely determined.*
2. *If* $X = Z$, *then* D *is determined up to linear equivalence.*

By above lemma, we sometimes use D to denote a pseudo-divisor triple.

Given a pseudo divisor D, we can associate a Cartier divisor C represents D. We define $[D] := [C]$ to be the Weil divisor associated to the pseudo-divisor D.

If (L, Z, s) and (L', Z', s') two pseudo-divisors, the sum $D + D' := (L \otimes L', Z \cup Z', s \otimes s')$. Similarly $-D = (L^{-1}, Z, 1/s)$. Fix $Z \subset X$ closed, the pseudo-divisor with support Z form a group.

Let $f : X' \to X$ be a morphism. $D = (L, Z, s)$ on X then we define the pullback

$$f^*D = (f^*L, f^{-1}(Z), f^*(s))$$

3.5 REMARK. The reason we use pseudo-divisor instead of Cartier divisor because Cartier divisor behaves badly with respect to pullback.

3.2 *Intersection with divisor*

3.6 DEFINITION. Let D be a pseudo-divisor on X, V be a k-dimensional subvariety of X. Then we define a class denoted by $D \cdot [V]$ or $D \cdot V$ in $A_{k-1}(|D| \cap V)$. As $j : V \to X$ inclusion. Then j^*D is a pseudo-divisor on V. Denote by $D \cdot [V]$ to be the Weil divisor class of $[j^*D]$ in $A_{k-1}(|V| \cap |D|)$.

For k-cycle $\alpha = \sum n_V[V]$ on X, the support of α written as $|\alpha|$ is union of the subvarieties V appearing with non-zero coefficient in α. For pseudo-divisor D on X each $D \cdot [V]$ is a class in $A_{k-1}(|D| \cap |\alpha|)$. We can define the intersection class

$$D \cdot \alpha \in A_k(|D| \cap |\alpha|)$$

by

$$D \cdot \alpha = \sum_V n_V D \cdot [V]$$

3.7 REMARK. The intersection class will be used to define two constructions

1. If $L = \mathcal{O}(D)$ a line bundle on X and $|D| = X$. Then $D \cdot \alpha$ will be $c_1(L) \cap \alpha$, the action of Chern class of L on α.
2. D is effective Cartier divisor on X, $i : D \hookrightarrow X$, then $D \cdot \alpha$ will be the Gysin pullback $i^*\alpha$.

3.8 PROPOSITION. *[Ful98, Proposition 2.3]*

1. *If* D *pseudo-divisor,* $\alpha \cdot \alpha'$ *k-cycle, then* $D \cdot (\alpha \cdot \alpha') = D \cdot \alpha + D \cdots \alpha'$.
2. *If* D, D' *pseudo-divisor,* α *k-cycle then* $(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha'$.
3. *(Projective formula)* $f : X' \to X$ *proper morphism* α *k-cycle on* X'

$$g : f^{-1}(|D| \cap f(|\alpha|)) \to |D| \cap |f(\alpha)|$$

Then $g_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha)$.

4. *Let* $f : X' \to X$ *flat,* α *k-cycle on* X.

$$g : f^{-1}(|D| \cap |\alpha|) \to |D| \cap |\alpha|$$

Then $f^*D \cdot f^*\alpha = g^*(D \cdot \alpha)$.

5. *If line bundle* $\mathcal{O}(D)$ *is trivial, and* α *a cycle, then* $D \cdot \alpha = 0$.

3.9 THEOREM. *[Ful98, Theorem 2.5] Let* D, D' *be Cartier divisor on n-dimensional variety. Then*

$$D \cdot [D'] = D' \cdot [D]$$

in $A_{n-2}(|D| \cap |D'|)$.

3.10 COROLLARY. *[Ful98, Corollary 2.6] Let* D *on* X, α *k-cycle rationally equivalent to* 0. *Then*

$$D \cdot \alpha = 0$$

in $A_{k-1}(|D|)$.

Proof. $\alpha = [\text{div } r]$, $D \cdot [\text{div } r] = \text{div}(r) \cdot [D] = 0$. □

3.11 DEFINITION. If D is a pesudo-divisor on X, Y closed subscheme. $\alpha \to D \cdot \alpha$ determines homomorphism

$$Z_k Y \to A_{k-1}(|D| \cap Y)$$

If α is rational equivalent to 0. Then

$$D \cdot \alpha = 0$$

So this gives

$$A_k Y \to A_{k-1}(|D| \cap Y)$$

3.12 COROLLARY. *Let* D, D' *be pseudo-divisor* X *for any k-cycle* α *on* X.

$$D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$$

3.13 DEFINITION. Let $D_1, \cdots, D_n$ be pseudo-divisor for k-cycle, we can define

$$D_1 \cdot D_n \cdot \alpha$$

More generally, $P(T_1, \cdots, T_n)$ homogeneous polynomial of degree d

$$p(D_1, \cdots, D_n) \cdot \alpha \in A_{k-d}(X)$$

If $n = k$, and $Y = |D_1| \cap \cdots \cap |D_K| \cap |\alpha|$ is complete. We define an intersection number $(D_1 \cdots D_n \cdot \alpha)_X$ by

$$(D_1 \cdot D_k \cdot \alpha)_X = \int_Y D_1 \cdots D_k \cdot \alpha$$

Similarly for P polynomial, we define

$$(P(D_1, \cdots, D_k) \cdot \alpha)_X = \int_Y P(D_1, \cdots, D_n) \cdot \alpha$$

3.14 EXAMPLE. $\pi : X \to \mathbb{A}^2$ be blow up of $\mathbb{A}^2$ at the origin $D \cdot D'$ be the inverse image of x-axis and y-axis. Then $D \cdot [D']$ and $D' \cdot [D]$ are well-defined cycles on $D \cap D' = E$. $D \cdot [D'] \neq D' \cdot [D] \in Z_0(X)$, but are equal in $A_0(X)$.

3.15 EXAMPLE. Let V be an irreducible surface and P is a singular point. Let $\pi : X \to V$ be a proper morphism $E = \pi^{-1}(P)$. Assume X regular, that π maps $X - E$ isomorphically onto $V - P$ and E connected. Then $(D \cdot D')_X < 0$ for any effective non-zero divisor D on X. (Because $(A \cdot B)_X \geq 0$ for surfaces if $|A| \cdot |B|$.)

3.16 EXAMPLE. $[x : y : z : t] \in \mathbb{P}^3$, X is a singular cone, deinfed by $z^2 = xy$. If $D = \{x = 0\}$, $L = \{x = z = 0\}$ and $L' = \{y = z = 0\}$. Let $P = \{[0 : 0 : 0 : 1]\}$. $[D] = 2[L]$ and $D \cdot [L'] = [P]$. It follows that there cannot be Cartier divisor D' on X with $[D'] = [L']$. $[L'] = [D']$ and $[P] = D \cdot [P'] = D' \cdot [P] = 2D' \cdot [L']$.

We define the Chern class of a line bundle to be operation

$$c_1(L) \cap [V] = [C]$$

Let V is k-variety. $L|_V$ is a Cartier divisor C determined up to linear equivalent [C] a Weil divisor.

If $L = \mathscr{O}_X(D)$, then $c_1(\mathscr{O}_X(D)) \cap \alpha = D \cdot \alpha$.

3.17 PROPOSITION. *[Ful98, Proposition 2.5]*

1. $c_1(L) \cap - : A_k X \to A_{k-1} X$
2. $c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$
3. $f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*\alpha$
4. $c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$
5. $c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$, $c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha$

Let D be an effective Cartier divisor and $i : D \to X$ inclusion. We define the Gysin map

$$i^* : Z_k X \to A_{k-1} D$$

determined by $i^*(\alpha) = D \cdot \alpha$.

3.18 PROPOSITION. *The Gysin map reduces to* $i^* : A_k X \to A_{k-1} D$.

REFERENCES

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