Non-ruledness via Hodge atoms

Nantao Zhang

Tsinghua University

Abstract

We apply the theory of Hodge atoms developed in [KKPY25] to show that cubic-threefolds and general cubic fourfolds are not ruled. We also discuss a possible derived category counterpart of atoms theory.

1 Introduction

Definition 1.1. *An algebraic variety X of dimension n is called* rational *if there is a birational equivalent $\mathbb{P}^n$ and is called* ruled *if it is birational equivalent to $Y \times \mathbb{P}^1$, where Y is some algebraic variety of dimension $n-1$. Since $\mathbb{P}^{n-1} \times \mathbb{P}^1$ is birational equivalent to $\mathbb{P}^n$ any rational variety is ruled.*

Determine whether an algebraic variety is rational is a long-standing problem in algebraic geometry. In [CG72], Clemens and Griffiths gives a proof on non-rationality of cubic threefolds via intermediate Jacobian. However, the non-rationality of general cubic fourfold remains a long standing problem until recent breakthrough by [KKPY25] via the theory of Hodge atoms. Also they give a new proof of non-rationality of cubic threefold.

On the other hand, the best result about non-ruledness of hypersurfaces is given by Kollár in [Kol95].

Theorem 1.2. *([Kol95, Theorem 4.1]) Let $X_d \subset \mathbb{P}^{n+1}$ be a very general hypersurface over $\mathbb{C}$. If $d \geq 2^r\lceil (n+3)/3 \rceil$, then X_d is not ruled.*

In particular, cubic threefolds and fourfolds ($d = 3$, $n = 3, 4$) are not included in his result. In this paper, we will prove the following.

Theorem 1.3. *(Theorem 3.3) Cubic threefolds and very general cubic fourfolds are not ruled.*

2 Hodge atoms

Here we give an overview of Hodge atoms developed in [KKPY25]. Let $\mathbb{K}$ be a non-archimedean field of characteristic 0. And $\mathbb{D}$ denote the germ at 0 in a $\mathbb{K}$-analytic unit disk with coordinate u.

Definition 2.1. *A* non-archimedean $\mathbb{K}$-analytic F bundle*, or* F-bundle *in short, is a triple $(\mathcal{H}, \nabla)/B$, such that*

1. *B is a smooth $\mathbb{K}$-analytic super variety or a germ of smooth $\mathbb{K}$-analytic super variety along an even closed smooth $\mathbb{K}$-analytic subvariety.*
2. *$\mathcal{H}$ is a $\mathbb{K}$-analytic super vector bundle over $B \times \mathbb{D}$.*
3. *∇ is a meromorphic flat connection on $\mathcal{H}$ with poles at most along $B \times \{0\}$, and for any vector field ξ on B, $\nabla_{u^2\partial u}$ and $\nabla_{u\xi}$ are regular.*

Let ξ be a vector field on B, then we define

$$\mu : T_B \to \mathrm{End}(\mathcal{H}|_{u=0})$$

to be the restriction of $\nabla_{u\xi}$ on $\mathcal{H}|_{u=0}$. And we define μ_b to be its further restriction to $\mathcal{H}_{(b,0)}$.

Definition 2.2. *An F-bundle $(\mathcal{H},\nabla)/B$ is called* maximal *(resp.* over-maximal*) at a geometric point $b\in B$ if there exists a vector $h\in\mathcal{H}_{(b,0)}$ such that*

$$\mathrm{ev}_h\circ\mu_b: T_{B,b}\to\mathcal{H}_{(b,0)}$$

is an isomorphism (resp. epimorphism).

And an F-bundle is maximal *(resp.* over-maixmal*) if it is maximal (resp. over-maixmal) everywhere.*

Let $(\mathcal{H},\nabla)/B$ be a maximal F-bundle, the *Euler vector field* is the unique even vector field Eu on B which under the action μ maps to endomorphism $\nabla_{u^2\partial_u}|_{\mathcal{H}|_{u=0}}$.

Now we construct the A-model F-bundle via Gromov-Witten theory. Let $\mathcal{K}=\overline{\mathcal{K}}$ an algebraic closed field of characteristic 0 and X a smooth projective $\mathcal{K}$-variety and $\beta\in\mathrm{CH}_1^{\mathrm{hom}}(X)$. We define $\overline{\mathcal{M}}_{0,n}(X,\beta)$ as the moduli of stable maps $\phi:(C,p_1,\cdots,p_n)\to X$ where

1. C is a connected nodal genus 0 curve.
2. $p_1,\cdots,p_n$ are smooth points of C.
3. $\phi_*[C]=\beta$ and if ϕ contracts a component of C to a point in X, then the number of marked points and nodes on the component ≥ 3.

There exists a virtual fundamental class on proper Deligne-Mumford stack $\overline{\mathcal{M}}_{0,n}(X,\beta)$ ([BF97])

$$[\overline{\mathcal{M}}_{0,n}(X,\beta)]_{\mathrm{vir}}\in\mathrm{CH}_{\delta(n,\beta)}(\overline{\mathcal{M}}_{0,n}(X,\beta))$$

Here

$$\delta(n,\beta)=n+(\dim X-3)+\int_\beta c_1(T_X)$$

is the virtual dimension. We define the Gromov-Witten cycle class $I_{n,\beta}(X)$ as the image

$$I_{n,\beta}(X)=\mathrm{ev}_*[\overline{\mathcal{M}}_{0,n}(X,\beta)]_{\mathrm{vir}}^{\mathrm{hom}}\in\mathrm{CH}^{\mathrm{hom}}_{\delta(n,\beta)}(X^{\times n})\otimes\mathbb{Q}$$

where

$$\mathrm{ev}:\overline{\mathcal{M}}_{0,n}(X,\beta)\to X^{\times n},(C,p_1,\cdots,p_n,\phi)\to(\phi(p_1),\cdots,\phi(p_n))$$

For simplicity, we just consider the case $\mathcal{K}=\mathbb{C}$ and $\mathbb{K}=\overline{\mathbb{Q}}((y^{\mathbb{Q}}))\supset k=\mathbb{Q}$ is the algebraic closed non-archimedean field of Puiseux series. Let $X/\mathcal{K}$ be a smooth projective variety, and $H^*(X)=H^*(X(\mathbb{C})^{\mathrm{an}},k)$. Let $N_1(X,\mathbb{Z})=\mathrm{CH}_1^{\mathrm{hom}}(X)$ the group of curve classes, and let $\mathrm{NE}(X,\mathbb{Z})\subset N_1(X,\mathbb{Z})$ the monoid of effective curve classes. We define

$$k[\mathrm{NE}(X,\mathbb{Z})]=\oplus_{\beta\in\mathrm{NE}(X,\mathbb{Z})}k\cdot q^\beta$$

with multiplication $q^{\beta_1}\cdot q^{\beta_2}=q^{\beta_1+\beta_2}$. And write $k[\![q]\!]=k[\![\mathrm{NE}(X,\mathbb{Z})]\!]$ for the completion with respect to ideal generated by all $q^\beta,\beta\neq 0$.

Let $\{T_i\}_{i=0,\cdots,r}$ be a homogeneous basis of $H^*(X)$, where $T_0=1\in H^0(X)$ and degree two basis are chosen to form two groups which spanning $H^2(X)_{\mathrm{alg}}$ and H^2_{trans} respectively. And (t_i) are coordinates with respect to the above basis. Then for $\gamma_1,\cdots,\gamma_n\in H^*(X)$ we define

$$\langle\gamma_1,\cdots,\gamma_n\rangle_\beta:=\int_{I_{n,\beta}(X)}\gamma_1\boxtimes\cdots\boxtimes\gamma_n$$

We define the H^*-valued genus 0 Gromov-Witten potential to be the formal power series

$$\Phi(q;t):=\sum_{n\geq 0,\beta\in\mathrm{NE}(X,\mathbb{Z})}\frac{q^\beta}{n!}\sum_{i_1,\cdots,i_n}\langle T_{i_1},\cdots,T_{i_n}\rangle_\beta t_{i_1}\cdots t_{i_n}\in k[\![q]\!][\![t_0,\cdots,t_r]\!]=:\mathrm{Nov}_X$$

Let $\mathcal{T}_{\mathbb{K}}(X):=\mathrm{NS}(X,\mathbb{Z})_{\mathrm{tf}}\otimes_{\mathbb{Z}}\mathbb{G}_{m,\mathbb{K}}$, where $\mathrm{NS}(X,\mathbb{Z})_{\mathrm{tf}}$ the torsion-free Néron-Severi group. Let $\{L_i\}$ be a sequence of ample line bundles, whose first Chern class ω_i form a basis of Néron-Severi group. We define $B_{X,q}$ to be the preimage of the ample cone under the valuation map of non-archimedean field $\mathcal{T}_{\mathbb{K}}(X)^{\mathrm{an}}\to\mathrm{NS}(X,\mathbb{R})$, with coordinates q_i corresponding to ω_i. Define $\mathcal{B}^{\mathrm{ev}}_{X,t}$ as product of analytic affine line corresponding to t_0 and unit polydisk with coordinates corresponding to $\deg T_i\in\{2,4,6,\cdots\}$. Define $B^{\mathrm{ev}}_{X,t}$ to be the product of analytic affine line corresponding to t_0 and unit polydisk with coordinates corresponding to $\deg T_i \in \{4, 6, 8, \cdots\}$ and $\deg T_i = 2$ and transcendental. Define B_X^{odd} as the super analytic variety whose underlying variety is a point and algebra of function is exterior algebra in the coordinates t_i for $\deg T_i \in \{1, 3, 5, \cdots\}$. Finally, we set $\mathcal{B}_X := B_{X,q} \times \mathcal{B}_{X,t}^{\mathrm{ev}} \times B_X^{\mathrm{odd}}$ $B_X := B_{X,q} \times B_{X,t}^{\mathrm{ev}} \times B_X^{\mathrm{odd}}$. Let $\mathcal{H} = H^*(X) \otimes_k \mathcal{O}_{\mathcal{B}_X \times \mathbb{D}}$ be the trivial vector bundle on $\mathcal{B}_X \times \mathbb{D}$. We have a canonical identification $\mathcal{H} = \oplus_{i=1}^r \mathcal{O}_{\mathcal{B}_X \times \mathbb{D}} \cdot T_i$. We can define quantum product

$$\star : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$$

by

$$\langle T_i \star T_j, T_k \rangle = \sum_{i=1}^{r} \langle \frac{\partial^3 \Phi}{\partial_{t_\alpha} \partial_{t_\beta} \partial_{t_i}} T^i, T_k \rangle$$

where T^i is the Poincaré dual of T_i.

Then we define the non-archimedean analytic quantum connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\mathcal{B}_X \times \mathbb{D}}[u, u^{-1}]$ to be

$$\begin{aligned} \nabla_{\partial_u} &= \partial_u - u^{-2}(\mathrm{Eu} \star (-)) + u^{-1} \frac{\mu - \dim X \cdot \mathrm{id}}{2} \\ \nabla_{\partial_{q_j}} &= \partial_{q_j} + u^{-1} q_j^{-1} (\omega_j \star (-)) \\ \nabla_{\partial_{t_i}} &= \partial_{t_i} + u^{-1} (T_i \star (-)) \end{aligned}$$

Here $\mu : \mathcal{H} \to \mathcal{H}$ is the degree operator $\mu = \oplus_{a=0}^{2 \dim X} a \cdot \mathrm{id}_{H^a(X)}$, and Eu is the Euler vector field for point $\gamma \in \mathcal{B}_X$ by

$$\mathrm{Eu}_\gamma = c_1(T_X) + \frac{\mu - 2\mathrm{id}}{2}(\gamma) \subset \mathcal{H}_\gamma$$

We define the *non-archimedean maximal A-model F-bundle* associated to X to be the restriction of $(\mathcal{H}, \nabla)/\mathcal{B}_X$ to B_X.

For a smooth complex variety X, we take $(\mathcal{H}, \nabla)/B_X$ to be the non-archimedean maximal A-model F-bundle. And let $\tilde{B} \to B_X$ be the ramified covering given by the spectrum of the Eu-action. $U_X \subset B$ the locus where the number of eigenvalues of the Eu-action is maximal, and $\tilde{U_X} := U_X \times_{B_X} \tilde{B}_{\mathrm{red}}$. The set of *local Hodge atoms* associated to X is $\pi_0(\tilde{U}_X)$ and the multiplicity of an $\alpha \in \pi_0(\tilde{U}_X)$ is defined to be the degree of the covering of corresponding connected component $\tilde{U}_{X,\alpha}$ over U_X.

The set HAtoms of all Hodge atoms of smooth projective variety is the quotient

$$\mathrm{HAtoms} := \left(\coprod_{[X]} \pi(\tilde{U}_X)/\operatorname{Aut}(X) \right) / \sim$$

where the union is taken over isomorphism class of complex smooth projective variety, and the equivalence relation is generated by following three elementary equivalences

1. [KKPY25, Section 5.2.3] If X_1 and X_2 are tow non-empty smooth projective varieties over $\mathcal{K}$. Then we have $\alpha \in \pi_0(\tilde{U}_{X_1})/\operatorname{Aut}(X_1)$ *disjoint union elementary equivalent* to its image α under the embedding

$$\pi_0(\tilde{U}_{X_1})/\operatorname{Aut}(X_1) \hookrightarrow \pi_0(\tilde{U}_{X_1 \coprod X_2})/\operatorname{Aut}(X_1 \coprod X_2)$$

2. [KKPY25, Section 5.2.4] Let X is a pure-dimensional smooth projective variety over $\mathcal{K}$, and $Z \subset X$ is a smooth projective subvariety of codimension ≥ 2. We denote by $\hat{X} = \mathrm{Bl}_Z X$ the blowup of X with center Z and $X' := X \coprod Z \coprod \cdots \coprod Z$ disjoint union of X and $(m-1)$-copies of Z. Then we have local atom $\alpha \in \pi_0(X)/\operatorname{Aut}$ to be *blowup elementary equivalent* to a local atom $\alpha' \in \pi_0(X')/\operatorname{Aut}(X')$ via the following correspondence

$$\begin{array}{ccccccc} & & & \pi_0(\tilde{\mathbb{U}}_{\hat{X}}) \cong \pi_0(\mathbb{U}_{X'}) & & & \\ & & \swarrow & & \searrow & & \\ & \pi_0(\tilde{U}_{\hat{X}}) & & & & \pi(\tilde{U}_X) \coprod \pi_0(\tilde{U}_Z)^{\coprod m-1} & \\ \swarrow & & & & & & \searrow \\ \pi_0(\tilde{U}_{\hat{X}})/\operatorname{Aut}(\hat{X}) & & & & & & \pi_0(\tilde{U}_{X'})/\operatorname{Aut}(X') \end{array}$$

Here $\mathbb{U}_{\hat{X}} \subset U_{\hat{X}}$ and $\mathbb{U}_{X'} \subset U_{X'}$ are subset over which the corresponding A-model F-bundle of $\hat{X}$ and X' coincides and $\tilde{\mathbb{U}}_{\hat{X}}$ and $\tilde{\mathbb{U}}_{X'}$ are their pullback to ramified covers. The existence of such $\mathbb{U}_{\hat{X}}$ and $\mathbb{U}_{X'}$ follows from Iritani's blow-up formula [Iri23].

3. [KKPY25, Section 5.2.5] Suppose X is a non-empty smooth projective variety over $\mathcal{K}$ and E is a vector bundle over X of rank ≥ 2. By the results of [IK23] and [HYZZ25], there exists a non-empty connected domain $\mathbb{U}_{\mathbb{P}(E)} \subset U_{\mathbb{P}(E)}$ and $\mathbb{U}_X \subset U_X$ such that $\mathbb{U}_{\mathbb{P}(E)} \cong \mathbb{U}_X^{\coprod r}$ such that the Euler operator are compatible. This gives a correspondence between local atoms of $\mathbb{P}(E)$ to the local atoms of $X^{\coprod r}$ and we say $\alpha \in \pi_0(\tilde{U}_{\mathbb{P}(E)})/\operatorname{Aut}(\mathbb{P}(E))$ is *Leray-Hirsch elementary equivalent* to the corresponding local atoms $\alpha' \in \tilde{U}_{X^{\coprod r}}/\operatorname{Aut}(X^{\coprod r})$.

3 Application to Cubic Hypersurfaces

By definition the Hodge atoms admits a natural filtration

$$\mathrm{HAtoms}_{\leq 0} \subset \mathrm{HAtoms}_{\leq 1} \subset \cdots$$

Our first result is an obvious improvement of [KKPY25, Proposition 5.17],

Proposition 3.1. *Let X be a smooth projective variety of dimension $d \geq 2$ over $\mathcal{K}$. Suppose we have a local Hodge atom α of X such that $\alpha \notin \mathrm{HAtoms}_{\dim \leq d-2}$. Then X can not be birationally equivalent to $Y \times \mathbb{P}^2_{\mathcal{K}}$, where Y is a variety of dimension $d-2$. In particular, X can not be birationally equivalent to $\mathbb{P}^d_{\mathcal{K}}$.*

Proof. Suppose X is birationally equivalent to $Y \times \mathbb{P}^2$ then by weak factorization theorem [Wlo02], there exists a series of blowups and blow-downs with smooth centers connecting X and $Y \times \mathbb{P}^2$. Since the centers must have codimension at least 2, and every local atom of $Y \times \mathbb{P}^2$ must belong to $\mathrm{HAtoms}_{\leq 2}$ by Leray-Hirsch elementary equivalence, every local atom of X also contains in $\mathrm{HAtoms}_{\leq 2}$ which is a contradiction. □

The following lemma gives an atom theoretic non-ruledness criterion.

Lemma 3.2. *Let X be a smooth projective variety of dimension $d \geq 2$ over $\mathcal{K}$. Suppose the number of local Hodge atom (counting by multiplicity) α of X such that $\alpha \notin \mathrm{HAtoms}_{\dim \leq d-2}$ is odd. Then X is not birationally equivalent to $Y \times \mathbb{P}^1_{\mathcal{K}}$, where Y is a variety of dimension $d-2$. That is X is not ruled.*

Proof. We argue by contradiction. If X is birational to $Y \times \mathbb{P}^1$. Then Hodge atom of X is equal to 2-copies of Hodge atoms of Y modulo $\mathrm{HAtoms}_{\leq \dim X - 2}$. Therefore, counting by multiplicity, the total number of local Hodge atoms α with $\alpha \notin \mathrm{HAtoms}_{\dim \leq d-2}$ must be odd. □

For cubic threefold, we can do better.

Theorem 3.3. *1. A cubic threefold is not ruled.*

2. A very general cubic threefold is not ruled.

Proof. Let X be a smooth cubic threefold. From calculation of [KKPY25, Example 6.21], we know that the atomic decomposition of X contains one indecomposable atom α corresponding to the eigenvalue 0. Also α cannot come from a Hodge atom of dimension ≤ 1. By Lemma 3.2, a cubic threefold is not ruled.

The second claim follows from calculation ofHodge atoms of very general cubic fourfolds in proof of [KKPY25, Theorem 6.8] and Lemma 3.2. □

4 Derived Category Counterpart

In [Kuz16], the author suggested a possible approach to prove the non-rationality via derived category method. Let $\mathcal{A}$ be a triangulated category, one can define the *geometric dimension* of $\mathcal{A}$ denoted by $\mathrm{gdim}(\mathcal{A})$ as the minimal integer n such that $\mathcal{A}$ can be realized as an admissible subcategory of a smooth projective variety of dimension n. And a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_n \rangle$ is called *maximal* if each component $\mathcal{A}_i$ is indecomposable. Let X be a projective variety of dimension n, and $D^b(X) = \langle \mathcal{A}_1, \cdots, \mathcal{A}_n \rangle$ to be a maximal semiorthogonal decomposition, then we "define" *Griffiths component* of X to be

$$\mathrm{Griff}(X) := \{\mathcal{A}_i \mid \mathrm{gdim}(\mathcal{A}_i) \geq n-1\}$$

If Griffiths component is well defined, i.e. it doesn't depend on the choice of maximal decomposition, then by weak factorization theorem and blow-up formula for derived category [BO95], one can use the non-emptiness of Griffiths component to show that a variety is not rational. However, as already mentioned in [Kuz16], the Griffiths component is not well defined, i.e. it depends on the choice of maximal decomposition. Here are some known counterexamples.

Example 4.1. *[Kuz13] Let*

$$Q = \left(\bullet \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bullet \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \bullet \mid \beta_1\alpha_2 = \beta_2\alpha_1 \right)$$

be the quiver. Then $D(Q) = \langle P_1, P_2, P_3 \rangle$ *where* P_i *are the projective module of i-th vertex. Also, there exists another exceptional object*

$$P = \left(k \underset{0}{\overset{\mathrm{id}}{\rightrightarrows}} k \underset{0}{\overset{\mathrm{id}}{\rightrightarrows}} k \right)$$

And $D^b(Q) = \langle P^\perp, P \rangle$*, and every indecomposable admissible subcategory of* $P^\perp$ *has geometric dimension* > 1*. However,* $D(Q)$ *can be realized as an admissible subcategory of a rational threefold* X *which is two step blow-up of* $\mathbb{P}^3$*.*

Example 4.2. *There are more examples about phantoms (admissible subcategory with a trivial Hochschild homology and a trivial Grothendieck group) and quasiphantoms (admissible subcategory with a trivial Hochschild homology and a finite Grothendieck group) [GO13]. Obviously,* $\mathrm{gdim}(\mathcal{A}) > 1$ *of a phantom or quasiphantom* $\mathcal{A}$*. Therefore [Kra24] gives a counterexample.*

Based on construction of Hodge atoms, we consider the following modification of the above idea, by introducing stability conditions [Bri07]. We first assume the following conjecture.

Conjecture 4.3. *Let* X *be a smooth projective variety. Then* $D^b(X)$ *has Bridgeland stability conditions.*

Let $\mathcal{T} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_n \rangle$ is called *maixmal polarized* if every $\mathcal{A}_i$ is indecomposable and there exists a Bridgeland stability on $\mathcal{A}_i$. Given a maximal polarized semiorthogonal decomposition, we define the *polarized Griffiths component* of X similarly, denoted by $\mathrm{pGriff}(X)$.

Conjecture 4.4. *Polarized Griffiths component is well defined, i.e. independent of choice of maximal polarized semiorthogonal decomposition.*

Remark 4.5. *In [HW25], it is shown that even polarized semiorthogonal decomposition doesn't have Jordan-Hölder property, but their example doesn't directly lead to the counterexample of above conjecture, because the components may still lie in some derived category of a variety of codimension* ≥ 2*.*

Although, we can not prove above conjecture, we give some evidence of above conjecture. First, we show that the above two counterexamples is no longer a counterexample for our new conjecture. In Example 4.1, the category $P^\perp$ doesn't have Bridgeland stability condition [HW25], therefore at least one of its indecomposable admissible subcategory has no semiorthogonal decomposition, otherwise the glueing [CP10] will produce stability condition on $P^\perp$. For Example 4.2, the phantom or quasiphantom category can not have Bridgeland stability condition by definition.

On the other hand, by [BMMS12] and [BLMS23], the Kuznetsov components of cubic threefold or fourfold have Bridgeland stability condition, and therefore the argument in [Kuz10] and [Kuz16] can still be applied to prove the non-rationality of cubic threefold and general cubic fourfold.

We also point out that a special case of DK conjecture [Huy06] that two birational smooth projective Calabi-Yau varieties are derived equivalent is a corollary of above two conjecture. The conjecture is known to be true for Calabi-Yau threefold by [Bri02].

Corollary 4.6. *If Conjecture 4.3 and Conjecture 4.4 holds, then every two birational Calabi-Yau variety are derived equivalent.*

Proof. Let X and Y be two birational Calabi-Yau variety. Since Calabi-Yau category is indecomposable and by Hochschild homology, they cannot be admissible subcategory of varieties of $\dim X - 2$. Therefore, we have

$$\{D^b(X)\} = \mathrm{pGriff}(X) = \mathrm{pGriff}(Y) = \{D^b(Y)\}$$

□

Remark 4.7. *The above reasoning have also been pointed out in [Hal24]. But our conjecture seems to be weaker than the grand program proposed there.*

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