LECTURE OF TORIC VARIETIES

- Toric varieties
  - Cones & Fans.
    - Compactness & properness.
    - Resolution
    - Orbits,
    - Divisors. & line bundles.
    - cohomology
    - Intersection theory.
    - Canonical divisor, Hirzebruch - Riemann - Roch
  - Polytopes :
  - Homogeneous coordinates.
  - moment map, symplectic reduction.
  - Gromov - Witten theory of toric varieties.
  - Kähler - Einstein metric of toric manifolds.

- Cones and Fans.

$$T_N = G := (\mathbb{G}_m)^n = \{ (t_1, \cdots, t_n) \mid t_i \in \mathbb{C}^* \}.$$

$$N = \mathbb{Z}^n = \left\{ \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \;\middle|\; b_j \in \mathbb{Z} \right\}.$$

$$M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) = \{ \vec{a} = (a_1, \cdots, a_n) \mid a_j \in \mathbb{Z} \}.$$

$\vec{a} \in M$, $\vec{b} \in N$,

$\chi^{\vec{a}} \in \mathrm{Hom}_{\text{alg gp}}(G, \mathbb{G}_m)$, $\lambda_{\vec{b}} \in \mathrm{Hom}_{\text{alg gp}}(\mathbb{G}_m, G)$

$$(\vec{a}, \vec{b}) = \sum_{i=1}^{n} a_i b_i \in \mathbb{Z}.$$

$$\chi^{\vec{a}}((t_1, \cdots, t_n)) := t_1^{a_1} \cdots t_n^{a_n}$$

$$\lambda_{\vec{b}}(t) := (t^{b_1}, t^{b_2}, \cdots, t^{b_n})$$

where $t, t_1, \cdots, t_n \in \mathbb{G}_m = \mathbb{C}^*$.

$$\begin{aligned} M &\xrightarrow{\sim} \mathrm{Hom}_{\text{alg gp}}(G, \mathbb{G}_m) \\ \vec{a} &\mapsto \chi^{\vec{a}} \end{aligned}$$

$$\begin{aligned} N &\xrightarrow{\sim} \mathrm{Hom}_{\text{alg gp}}(\mathbb{G}_m, G) \\ \vec{b} &\mapsto \lambda_{\vec{b}} \end{aligned}$$

$$\chi^{\vec{a}} \lambda_{\vec{b}}(t) = t^{(\vec{a}, \vec{b})} \quad \text{for all } t \in G_m = \mathbb{C}^*.$$

A rational polyhedral cone $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is

$$\sigma = \{ \textstyle\sum_{i=1}^{s} \lambda_i u_i : \lambda_i \geq 0 \}$$

where $u_1, \cdots, u_s \in N$.

- σ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$.
- $\dim(\sigma) :=$ dim (of linear space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$)
- The dual cone $\sigma^\vee$ is

$$\sigma^\vee = \{ m \in M_{\mathbb{R}} \mid \langle m, v \rangle \geq 0 \text{ for all } v \in \sigma \}.$$

- $(\sigma^\vee)^\vee = \sigma$.
- A face $\tau \subset \sigma$ is

$$\tau = \{ v \in \sigma \mid \langle m, v \rangle = 0 \} \subset \sigma$$

for some $m \in M \cap \sigma^\vee$.

A facet is a face of codim 1.

- $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$,

$$\dim \tau + \dim(\sigma^\vee \cap \tau^\perp) = n = \dim N_{\mathbb{R}}$$

Prop : (Gordon's lemma)

If σ is a rational polyhedral cone, then

$S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.

$\therefore$ $\mathbb{C}[S_\sigma]$ is a f.g. $\mathbb{C}$-alg.

identified to be $\chi^{\vec{a}} \in \mathrm{Hom}_{\mathrm{alg\,gp}}(T_N, \mathbb{C}^*)$

$U_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$.

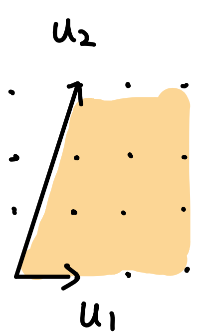
Example :

(i) • $\sigma = \{0\}$,

$\sigma^\vee = M$.

$\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, X_1^{-1}, \cdots, X_n, X_n^{-1}]$

$U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma] = (\mathbb{C}^*)^n = T_N$.

(ii) 

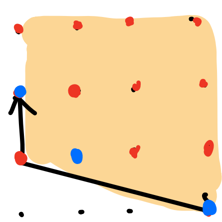
$u_1 = e_1$, $u_2 = e_1 + 3e_2$.

$m = m_1 e_1^\vee + m_2 e_2^\vee$

$\langle m, u_1 \rangle = m_1 \geq 0$.

σ

$\langle m, u_2 \rangle = m_1 + 3m_2 \geq 0$.



$\therefore$ $\sigma^\vee = \mathrm{Cone}(3e_1^\vee - e_2^\vee, e_2^\vee)$

$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi_1^3 \chi_2^{-1}, \chi_2]$.

$\sigma^\vee \qquad = \mathbb{C}[\chi_1^3\chi_2^{-1}, \chi_1, \chi_2]$

$\qquad\qquad\qquad u \quad , \quad v \quad , \quad w$

$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = V(uw - v^3)$. $\quad A^2$-singularity.

(iii) $\sigma = \sum_{i=1}^{k} \mathbb{R}_{\geq 0} e_i$

$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$

Properties of U_σ:

(i) Each ring $A_\sigma = \mathbb{C}[S_\sigma]$ is integrally closed.

i.e. U_σ is normal. In particular,

$$\dim(\mathrm{sing}\, U_\sigma) \leq \dim U_\sigma - 2.$$

(ii) Cohen-Macauley $\Rightarrow$ We can use Serre duality.

(iii) nonsingular / smooth

$\Leftrightarrow$ σ is generated by part of a basis for N

$\Leftrightarrow$ $U_\sigma \simeq \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$ $\qquad k = \dim \sigma$.

$\sigma \subseteq N_{\mathbb{R}}$ strongly convex rational polyhedral cone

$\lim_{t\to 0} \lambda_{\vec{b}}(t)$ exists in U_σ $\Leftrightarrow$ $\lim_{t\to 0} \chi^{\vec{a}} \lambda_{\vec{b}}(t)$ exists in $\mathbb{C}$

for all $\vec{a} \in S_\sigma$.

§ Toric Varieties $X(\Sigma)$ $\left(= \mathbb{P}_\Sigma = \mathbb{P}_\Delta\right)$ $\left(\Sigma = \text{normal cone of } \Delta\right)$

A fan Σ in $N_{\mathbb{R}}$ consists of a finite collection of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying

- If $\sigma \in \Sigma$, then every face of σ is also in Σ.
- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

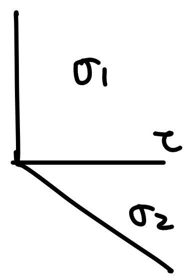
$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}$ is the support of Σ.

$\Sigma(d) := \{ d\text{-dimensional cones of } \Sigma \}$.

- $\tau \subset \sigma \Rightarrow \tau^\vee \supset \sigma^\vee \Rightarrow S_\tau \supseteq S_{\sigma^\vee}$

$\Rightarrow U_\tau \subset U_\sigma$

$\therefore$



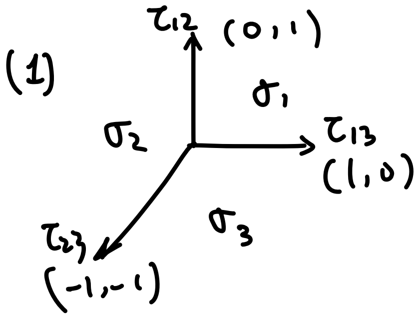
$\rightsquigarrow U_\tau \subseteq U_{\sigma_1} \cap U_{\sigma_2}$

Lemma: If σ_1 and σ_2 are cones that intersect in a common face, then the diagonal map $U_{\sigma_1 \cap \sigma_2} \to U_{\sigma_1} \times U_{\sigma_2}$ is a closed embedding. In particular, $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$.

$\rightsquigarrow X(\Sigma) = \bigcup_{\sigma \in \Sigma} U_\sigma$.

Example :

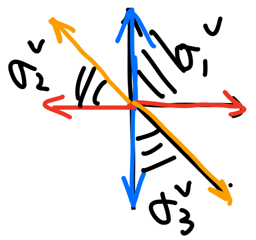
(1)



$$U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[x_1, x_2] \simeq \mathbb{A}^2$$

$$U_{\sigma_2} = \operatorname{Spec} \mathbb{C}[x_1^{-1}x_2, x_1^{-1}] \simeq \mathbb{A}^2$$

$$U_{\sigma_3} = \operatorname{Spec} \mathbb{C}[x_1x_2^{-1}, x_2^{-1}] \simeq \mathbb{A}^2$$



$$U_{\tau_{12}} = \operatorname{Spec} \mathbb{C}[x_1, x_1^{-1}, x_2] \simeq \mathbb{C}^* \times \mathbb{C}$$

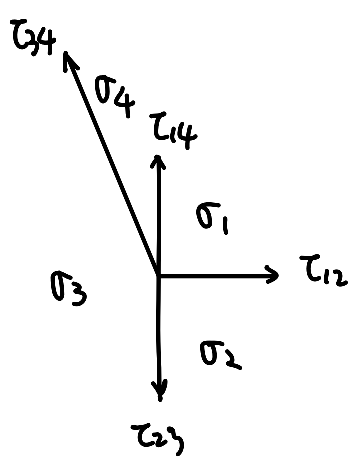
$$U_{\tau_{13}} = \operatorname{Spec} \mathbb{C}[x_2, x_2^{-1}, x_1] \simeq \mathbb{C}^* \times \mathbb{C}$$

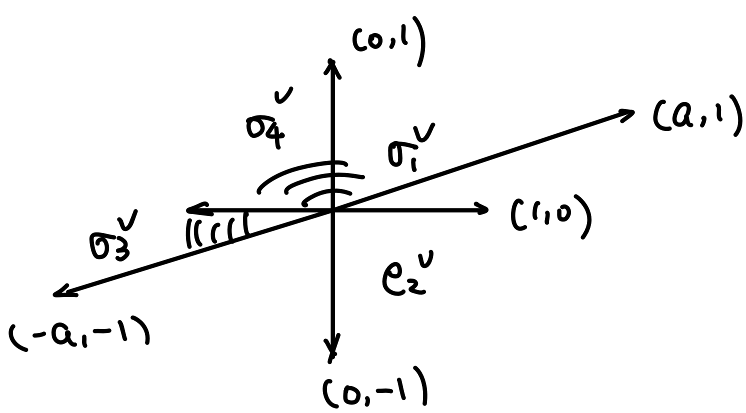
$$U_{\tau_{23}} = \operatorname{Spec} \mathbb{C}[x_1^{-1}x_2, x_1x_2^{-1}, x_1^{-1}x_2^{-1}] \simeq \mathbb{C}^* \times \mathbb{C}$$

$$\begin{tikzcd} & U_{\sigma_1} \arrow[dl, leftrightarrow] \arrow[dr] & \\ U_{\sigma_2} \arrow[rr, leftrightarrow] & & U_{\sigma_3} \end{tikzcd} \qquad \begin{tikzcd} & (x_1, x_2) \arrow[dl] \arrow[dr] & \\ (x_1^{-1}, x_1^{-1}x_2) & & (x_1x_2^{-1}, x_2^{-1}) \end{tikzcd} \qquad \begin{tikzcd} & \left(\frac{T_0}{T_1}, \frac{T_2}{T_1}\right) \arrow[dl] \arrow[dr] & \\ \left(\frac{T_1}{T_0}, \frac{T_2}{T_0}\right) & & \left(\frac{T_0}{T_2}, \frac{T_1}{T_2}\right) \end{tikzcd}$$

$$(T_0 : T_1 : T_2) \quad x = \frac{T_0}{T_1}, \quad y = \frac{T_2}{T_1}, \quad x^{-1} = \frac{T_1}{T_0}, \quad y^{-1} = \frac{T_1}{T_2}$$

(2)





$U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[x_1, x_2]$, $\quad U_{\sigma_2} = \operatorname{Spec} \mathbb{C}[x_1, x_2^{-1}]$

$U_{\sigma_3} = \operatorname{Spec} \mathbb{C}[x_1^{-1}, x_1^{-a}x_2^{-1}]$, $\quad U_{\sigma_4} = \operatorname{Spec} \mathbb{C}[x_1^{-1}, x_1^{a}x_2]$

$U_{\tau_{12}} = \operatorname{Spec} \mathbb{C}[x_2, x_2^{-1}, x_1]$, $\quad U_{\tau_{14}} = \operatorname{Spec} \mathbb{C}[x_1, x_1^{-1}, x_2]$

$U_{\tau_{34}} = \operatorname{Spec} \mathbb{C}[x_1^{a}x_2, x_1^{-a}x_2^{-1}, x_1^{-a}x_2]$, $\quad U_{\tau_{23}} = \operatorname{Spec} \mathbb{C}[x_1, x_1^{-1}, x_2^{-1}]$

$$\begin{tikzcd}
U_{\sigma_1} \arrow[rrr] \arrow[ddd] & & & U_{\sigma_2} \arrow[ddd] \\
& (x_1, x_2) \arrow[r, mapsto] \arrow[d, mapsto] & (x_1, x_2^{-1}) \arrow[d, mapsto] & \\
& (x_1^{-1}, x_1^{a}x_2) \arrow[r, mapsto] & (x_1^{-1}, x_1^{-a}x_2^{-1}) & \\
U_{\sigma_4} \arrow[rrr] & & & U_{\sigma_3}
\end{tikzcd}$$

$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(a))$

$([u, v], [z, w])$

$x_1 = \frac{v}{u}$, $\quad x_2 = \frac{z}{w}$

$\mathcal{O}(-1)$: transition map:

$$\varphi_{\lambda\mu}((z^0 : \cdots : z^\mu : \cdots : z^m)) = \frac{z^\lambda}{z^\mu}.$$

$U_u = \{u \neq 0\}$, $\quad U_v = \{v \neq 0\}$.

$z^\mu = u \qquad z^\lambda = v$

$\therefore \quad \mathcal{O} \oplus \mathcal{O}(a)$ transition map:

$$\varphi_{\lambda\mu} : \quad U_u \times \mathbb{A}^2 \longrightarrow U_v \times \mathbb{A}^2$$

$$\left(\frac{v}{u}, z, w\right) \longmapsto \left(\frac{u}{v}, z, \left(\frac{v}{u}\right)^{-a} w\right)$$

$$\left[1, \left(\frac{v}{u}\right)^{-a} \frac{w}{z}\right]$$

$$\left[\frac{v^a z}{u^a w}, 1\right]$$

$$
\begin{tikzcd}
U_{\sigma_1}=U_{u,w} \arrow[rrr] \arrow[ddd, "\varphi_{\lambda\mu}"'] & & & U_{u,z}=U_{\sigma_2} \arrow[ddd, "\varphi_{\lambda\mu}"] \\
& \left(\frac{v}{u},\frac{z}{w}\right) \arrow[r, mapsto] \arrow[d, mapsto] & \left(\frac{v}{u},\frac{w}{z}\right) \arrow[d, mapsto] & \\
& \left(\frac{u}{v},\frac{v^a z}{u^a w}\right) \arrow[r, mapsto] & \left(\frac{u}{v},\frac{v^{-a} w}{u^{-a} z}\right) & \\
U_{\sigma_4}=U_{v,w} \arrow[rrr] & & & U_{v,z}=U_{\sigma_3}
\end{tikzcd}
$$

(3) $\mathbb{P}(\mathcal{O}(a_1)\oplus\cdots\oplus\mathcal{O}(a_r)) \to \mathbb{P}^n$

Let N be the lattice of rank $r+n-1$ generated by vectors $w_1,\cdots,w_r$ and $v_0,\cdots,v_n$ with relations

$$w_1+\cdots+w_r=0, \quad v_0+\cdots+v_n = a_1w_1+\cdots+a_rw_r.$$

(4) $\mathbb{P}(r_0,\cdots,r_n)$

Let $v_0,\cdots,v_n \in N$ such that

$$r_0v_0+r_1v_1+\cdots+r_nv_n=0 \quad \text{in } N,$$

Let $\Sigma \subseteq N_{\mathbb{R}}$ be the fan generated by all the cones given by all subsets of $\{v_0,\cdots,v_n\}$.

$$X(\Sigma) \cong \mathbb{P}(r_0,\cdots,r_n).$$

Morphism: $\Sigma' \subseteq N'$, $\Sigma \subseteq N$

$\varphi: N' \to N$ homo of lattices

$\forall \sigma' \in \Sigma', \exists \sigma\in\Sigma$ s.t. $\varphi(\sigma')\subseteq\sigma$

$\Rightarrow U_{\sigma'} \longrightarrow U_\sigma \subseteq X(\Sigma) \rightsquigarrow X(\Sigma') \xrightarrow{\varphi_*} X(\Sigma)$

Prop : fan Σ $\longleftrightarrow$ geometry of $X(\Sigma)$

- $X(\Sigma)$ is compact (i.e. complete) $\Longleftrightarrow$ $|\Sigma| = N_{\mathbb{R}}$.

- Let $\varphi : N' \to N$ be a homomorphism of lattice that maps a fan Σ' to Σ, then

  $\varphi_* : X(\Sigma') \to X(\Sigma)$ is proper iff $\varphi^{-1}(|\Sigma|) = |\Sigma'|$.

- X is smooth $\Longleftrightarrow$ Every cone σ in Σ is generated by part of $\mathbb{Z}$ - basis of N.

  Such fan Σ is called smooth.

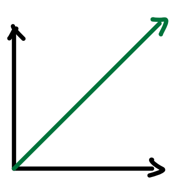
- X is an orbifold $\Longleftrightarrow$ the generators of every cone in Σ are linearly independent / $\mathbb{R}$.

  Such Σ, X are simplicial.

- Resolution of Singularities.

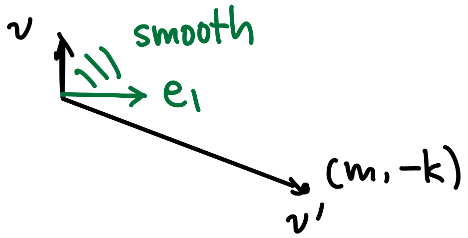
  Refine Δ s.t. each cone has unit volumes

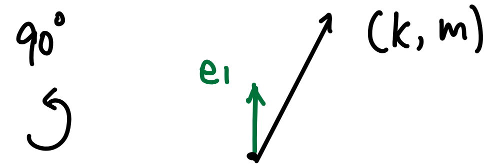
(1) Blow up :



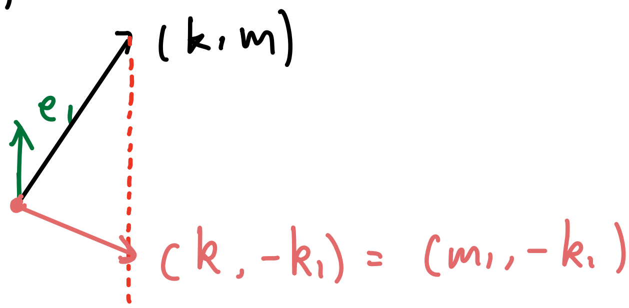
$$\begin{array}{ccc} X(\Delta') & \longrightarrow & X(\Delta) \\ \parallel & & \parallel \\ \mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & \mathbb{A}^2 \end{array}$$

(2) $v = e_2$, $v' = me_1 - ke_2$. $0 < k < m$. $\gcd(k, m) = 1$.





$$\xrightarrow{\exists \begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix}}$$



$m_1 = k$, $k_1 = a_1 k - m$ for some $a_1 \geq 2$.

$k_1 = 0 \iff$ smooth cone.

$$\frac{m}{k} = a_1 - \frac{k_1}{m_1} = a_1 - \frac{1}{m_1/k_1}$$

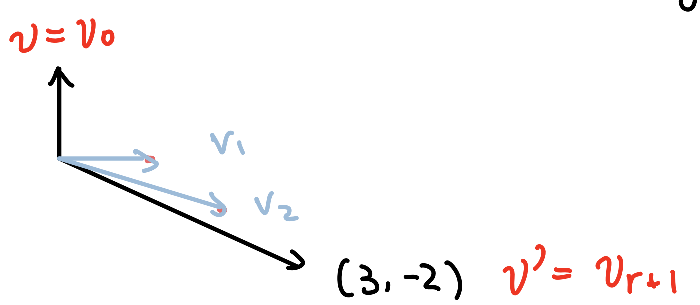
$$= a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_r}}} \quad \text{with } a_i \geq 2.$$

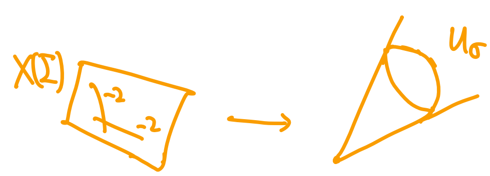
Rmk: $L(p,q)$: $\frac{p}{q} = a_1 - \cdots$ surgery

(3) (2) is equivalent to find generator of on N.

e.g. 0) $\frac{3}{2} = 2 - \frac{1}{2}$

$a_1 = 2$, $a_2 = 2$.





Relationship : add r $v_1, \cdots, v_r$, & $v_0 := v$, $v_{r+1} := v'$.

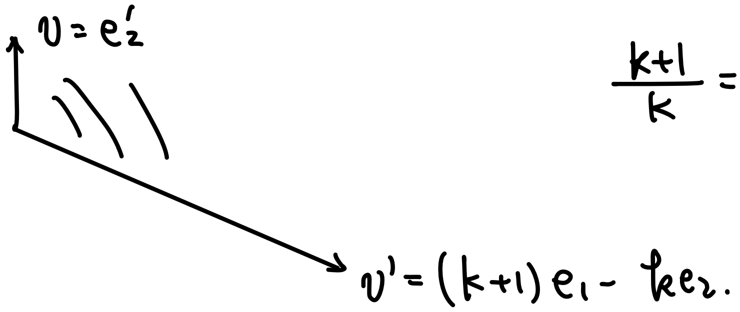
$$a_i v_i = v_{i-1} + v_{i+1} \quad (i = 1, \cdots, r.)$$

exceptional divisors $E_i \simeq \mathbb{P}^1$.

with self-intersection $E_i \cdot E_i = -a_i$. (explain later) intersection thy).

e.g. (1) A_k - singularities.

k points.



$$\frac{k+1}{k} = 2 - \cfrac{1}{\cfrac{k}{k-1}} = 2 - \cfrac{1}{2 - \cfrac{1}{2 \cdots}}$$

$\underbrace{\qquad\qquad}_{k \text{ terms.}}$

$X(\Sigma)$

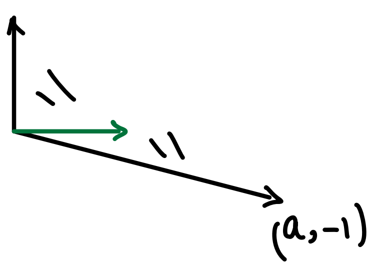
$$\downarrow$$

$$\mathbb{C}^2/\mathbb{Z}_{k+1} \simeq U_\sigma = \operatorname{Spec} \frac{\mathbb{C}[Y_1, Y_2, Y_3]}{Y_3^{k+1} - Y_1 Y_2}$$

$\mathbb{Z}_{k+1} \curvearrowright \mathbb{C}^2$

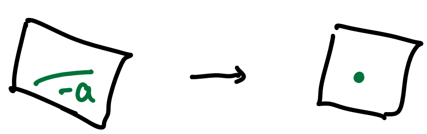
$\xi \qquad (\xi u, \xi^{-1} v)$

e.g. (2)



$$(a, -1) + (0, 1) = a \cdot (1, 0)$$

$$X(\Sigma) = \mathcal{O}_{\mathbb{P}^1}(-a)$$



(4) Toric flips and flops. (explain later).

flip: **Definition 6.12** (Log flip) Let $(X/Z, B)$ be a lc pair and $f\colon X \to Y/Z$ the contraction of a $K_X + B$-negative extremal ray of small type. The log flip of this flipping contraction is a diagram

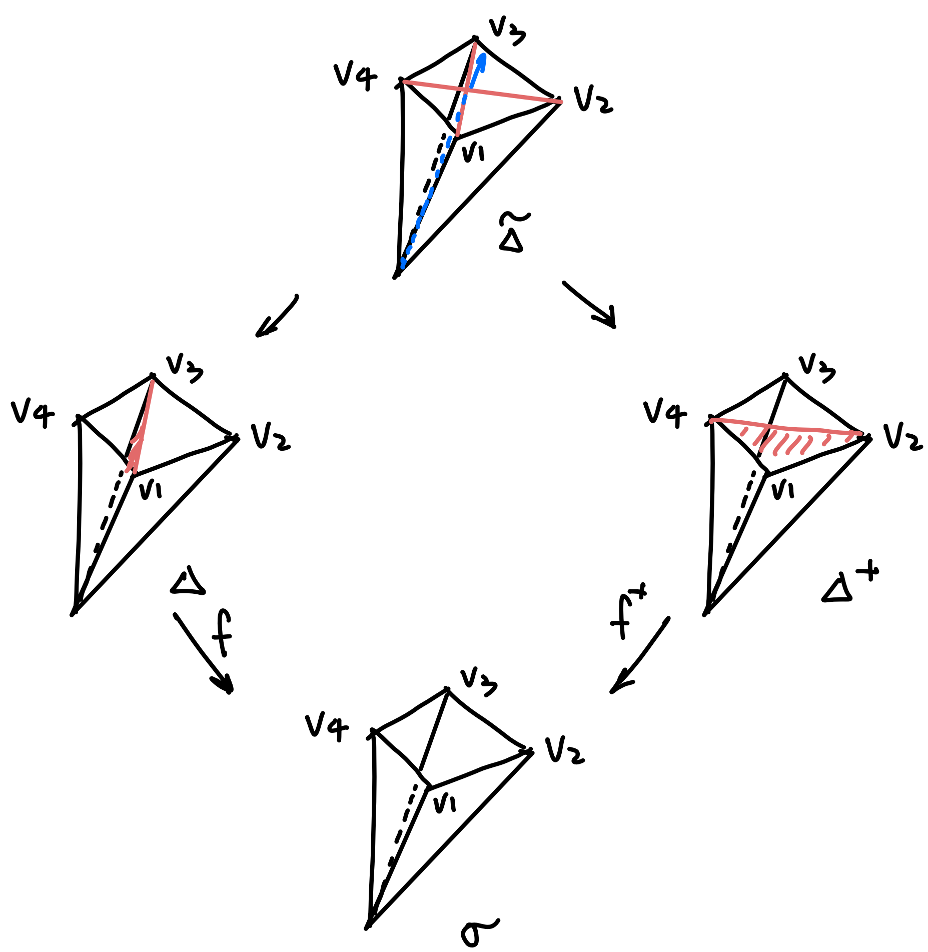
$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

such that

- X^+ is a normal variety, projective$/Z$,
- f^+ is a small projective birational contraction$/Z$,
- $-(K_X + B)$ is ample over Y (by assumption), and $K_{X^+} + B^+$ is ample over Y where B^+ is the birational transform of B.

flop: $-(K_X + B)$ is trivial over Y.

$N = \mathbb{Z}^3$, $\sigma \in N_{\mathbb{R}}$ generated by $v_1, \dots, v_4 \in N$.



$X(\tilde{\Delta})$ divisorial contraction.

$\swarrow \qquad \searrow$

birational

$$\begin{tikzcd} X(\Delta) \arrow[rr, dashed, "\mathrm{bir}"] \arrow[dr, "f"'] & & X(\Delta^+) \arrow[dl, "f^+"] \\ & X_\sigma & \end{tikzcd}$$

small contraction.

f contracts $V(\langle v_1, v_3\rangle) =: C$

f^+ contracts $V(\langle v_2, v_4\rangle) =: C^+$.

$K_{X(\Delta)} \cdot C < 0$: flip

$K_{X(\Delta)} \cdot C = 0$: flop.

$\exists\ a_1, a_2, a_3, a_4$ s.t.

$$a_1 v_1 + a_3 v_3 = a_2 v_2 + a_4 v_4$$

$a_1 = a_2 = a_3 = a_4 = 1$: flop case.

§ Orbits.

Torus action : if σ cone in N, T_N acts on U_σ

$$T_N \times U_\sigma \longrightarrow U_\sigma$$

- a point $t \in T_N \iff$ map $M \to \mathbb{C}^*$ of group
- $x \in U_\sigma \iff$ map $S_\sigma \to \mathbb{C}$ of semigp.
- $t \cdot x$: $S_\sigma \to \mathbb{C}$

$$u \mapsto t(u)x(u).$$

- alg map: $\mathbb{C}[S_\sigma] \to \mathbb{C}[M] \otimes \mathbb{C}[S_\sigma]$

$$\chi^u \mapsto \chi^u \otimes \chi^u$$

$\sigma = \{0\}$ usual product in T_N.

$$\begin{tikzcd} T_N \times X(\Delta) \arrow[r] & X(\Delta) \\ T_N \times T_N \arrow[u, equal] \arrow[u, hook] \arrow[r] & T_N \arrow[u, hook] \end{tikzcd}$$

σ: cone in N

The distinguished point x_σ:

$$S_\sigma = \sigma^\vee \cap M \longrightarrow \{1, 0\} \subset \mathbb{C}^* \cup \{0\} = \mathbb{C}.$$

$$u \mapsto \begin{cases} 1 & \text{if } u \in \sigma^\perp \\ 0 & \text{else}. \end{cases}$$

O_σ = orbit containing x_σ.

$\cong (\mathbb{C}^*)^{n-k}$ $\quad (\dim \sigma = k)$.

$V(\sigma)$ = orbit closure.

Prop: (i) $U_\sigma = \bigsqcup_{\tau < \sigma} O_\tau$

(ii) $V(\tau) = \bigsqcup_{\gamma > \tau} O_\gamma$

(iii) $O_\tau = V(\tau) \setminus \bigcup_{\gamma \gneq \tau} V(\gamma)$

Cohomology:

Lma: (i) σ : n-dim'l cone, then U_σ is contractible.

(ii) σ: k-dim'l, then $O_\sigma \subset U_\sigma$ is a deformation retract.

(iii) $\exists$ canonical isom $H^i(U_\sigma;\mathbb{Z}) \cong \Lambda^i(M(\sigma))$ where $M(\sigma) = \sigma^\perp \cap M$.

$$E_1^{p,q} = \bigoplus_{i_0<\cdots<i_p} H^q(U_{i_0}\cap\cdots\cap U_{i_p}) \Rightarrow H^{p+q}(X)$$

(a) $U_i = U_{\sigma_i}$, σ_i maximal cones of Σ.

$$E_1^{p,q} = \bigoplus_{i_0<\cdots<i_p} \Lambda^q M(\sigma_{i_0}\cap\cdots\cap\sigma_{i_p}) \Rightarrow H^{p+q}(X(\Sigma))$$

$$\begin{aligned}
\chi(X(\Sigma)) &= \sum_{p,q} (-1)^{p+q} \operatorname{rank} E_1^{p,q} \\
&= \sum_{p,q} (-1)^{p+q} \sum_{i_0<\cdots<i_p} \operatorname{rank} \Lambda^q(M(\sigma_{i_0}\cap\cdots\cap\sigma_{i_p}) \\
&= \sum_{i_0<\cdots<i_p} (-1)^p \sum_q (-1)^q \operatorname{rank} \Lambda^q M(\sigma_{i_0}\cap\cdots\cap\sigma_{i_p})
\end{aligned}$$

lemma: $\displaystyle\sum_q (-1)^q \operatorname{rank} \Lambda^q M(\tau) = \begin{cases} 0 & \dim\tau < n \\ 1 & \dim\tau = n. \end{cases}$

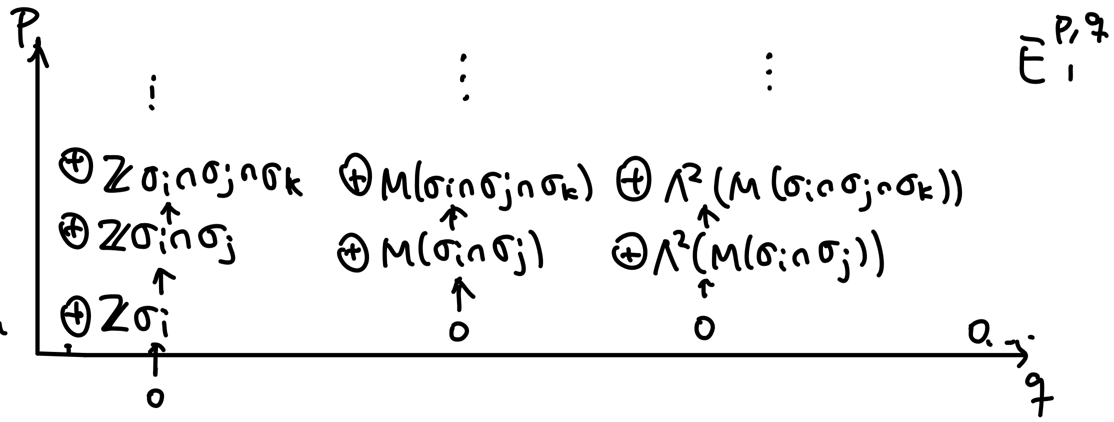
$= \#$ n-dim'l cones in Δ.

Prop: Assume all maximal cones in Δ is n dim'l.

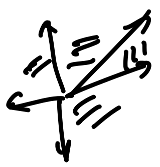
Since U_{σ_i} contractible, $E_1^{0,q} = 0$ for $q \geq 1$.

In addition, $E_1^{\bullet,0}$ is

$$0 \to \bigoplus_i \mathbb{Z}\sigma_i \longrightarrow \bigoplus_{i<j} \mathbb{Z}_{\sigma_i \cap \sigma_j} \longrightarrow \bigoplus_{i<j<k} \mathbb{Z}_{\sigma_i \cap \sigma_j \cap \sigma_k} \to \cdots$$

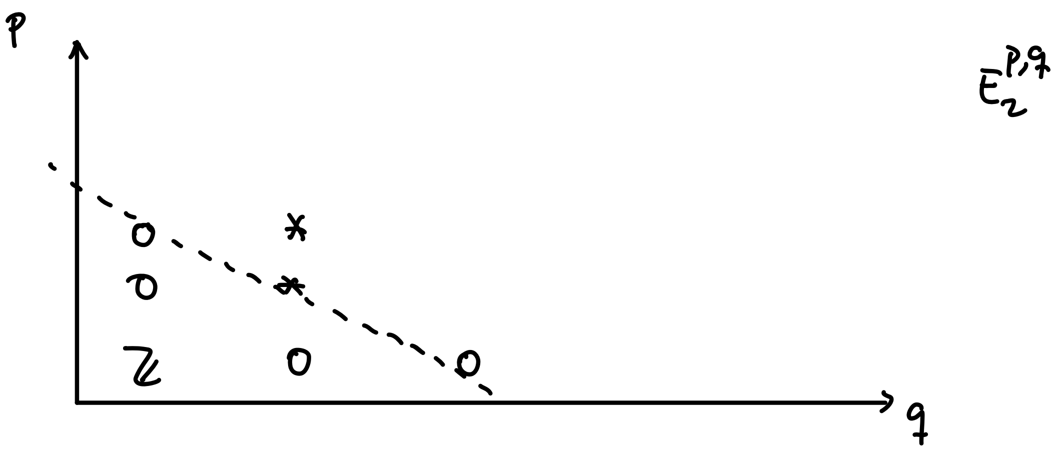


$E_1^{\bullet,0}$: cochain complex of a simplex.



$d' : E_1^{p,q} \longrightarrow E_1^{p+1,q}$ $\quad d_r^{pq} : E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$

$E_2^{p,0} = 0$ for $p \geq 1$



$$H^2(X(\Sigma)) = E_\infty^{1,1} = E_2^{1,1} = \operatorname{Ker}\Big(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \longrightarrow \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)\Big)$$

$\left(\begin{array}{l} \vec{x} : S_\sigma \to \mathbb{C} \\ \longleftrightarrow \text{ point in } U_\sigma. \end{array} \right)$ $\quad \chi^u(\vec{x}) = \vec{x}(u)$

$\langle \chi^u, D_i \rangle = \langle u, \tau_i \rangle = 0.$

$\forall\, u \in M(\sigma) = \sigma^\perp \cap M$ gives a nonvanishing section χ^u on U_σ. $\Rightarrow u_{ij} \in \operatorname{Ker}\big(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \to \cdots\big)$ means u_{ij} on

$U_{\sigma_i \cap \sigma_j}$ satisfying cocycle condition

$$\chi^{u_{ij}} \chi^{u_{jk}} \chi^{u_{ki}} = \chi^{u_{ij}+u_{jk}-u_{ik}} = 1.$$

$$H^2(X(\Sigma)) \xleftrightarrow{1:1} \text{line bundles}$$

first chern class.

Theorem 12.3.11. *If X_Σ is complete and simplicial, then $E_2^{p,q} = 0$ when $p \neq q$ in the spectral sequence* (12.3.11). *Thus:*

(a) $H^{2k+1}(X_\Sigma, \mathbb{Q}) = 0$ *for all k.*

(b) $H^{2k}(X_\Sigma, \mathbb{Q}) \simeq E_2^{k,k}$ *for all k.*

§ T-divisors.

A Cartier divisor $D = \{$ rational function $f_\alpha \neq 0$ on $U_\alpha \}$.

$\mathcal{O}(-D) :=$ sheaf of rational functions generated by (f_α, U_α)

$\mathcal{O}(D) :=$ ... $(\frac{1}{f_\alpha}, U_\alpha)$.

Transition functions: $U_\alpha \xrightarrow{\frac{f_\alpha}{f_\beta}} U_\beta$

$$\frac{1}{f_\alpha} \mapsto \frac{1}{f_\beta}.$$

$$\begin{aligned} CaCl(X(\Sigma)) &\longrightarrow Cl(X(\Sigma)) \\ D &\mapsto [D] = \sum_{\operatorname{cod}(V,X)=1} \operatorname{ord}_V(D) \cdot V \end{aligned}$$

$\operatorname{ord}_V(D) =$ order of vanishing of an equation for D in the local ring along V.

DVR because X is normal.

$X = X(\Sigma)$, $T = T_N$

T_N- stable subvarieties codim 1 $\Longleftrightarrow$ edges $\tau_1, \cdots, \tau_d$

$D_i = V(\tau_i)$.

v_i = first lattice point met along τ_i.

T- Weil divisors $= \{ \sum a_i D_i \mid a_i \in \mathbb{Z} \}$.

- T- Cartier divisors.

(a) affine case $X = U_\sigma$. $\dim \sigma = n$.

D = T- stable divisor with $I = \Gamma(X, \mathcal{O}(D))$.

lemma : I is generated by χ^u for $u \in \sigma^\vee \cap M$.

i.e. $D = \mathrm{div}(\chi^u)$ for some unique $u \in M$.

(b) lemma : Let $u \in M$, v = first lattice along an edge τ. Then $\mathrm{ord}_{V(\tau)}(\mathrm{div}(\chi^u)) = \langle u, v \rangle$

$$[\mathrm{div}(\chi^u)] = \sum_i \langle u, v_i \rangle D_i$$

(c) $X = U_\sigma$, $\underline{\dim \sigma < n}$

T- Cartier divisor on U_σ is of the form $\mathrm{div}(\chi^u)$ for some $u \in M$, but not unique

$$\mathrm{div}(\chi^u) = \mathrm{div}(\chi^{u'}) \Longleftrightarrow u - u' \in M(\sigma) = \sigma^\perp \cap M.$$

$\Leftrightarrow \exists!$ element in $M/M(\sigma)$.

(d) $X(\Sigma)$: D = T- Cartier divisor

$D|_{U_\sigma} = \operatorname{div} \chi^{u(\sigma)} \xleftrightarrow{1:1} u(\sigma) \in M/M(\sigma)$ for each σ.

glue together:

$$\{T\text{- Cartier divisor}\} \xleftrightarrow{1:1} \varprojlim M/M(\sigma)$$
$$= \ker\Big(\bigoplus_i M/M(\sigma_i) \longrightarrow \bigoplus_{i<j} M/(\sigma_i \cap \sigma_j)\Big)$$

Lma: A Weil divisor $D = \sum a_i D_i$ is Cartier iff

for each (maximal) cone σ, $\exists\, u(\sigma) \in M$ such that

for all $v_i \in \sigma$, $\langle u(\sigma), v_i \rangle = -a_i$.

$$\left(\Leftrightarrow \quad \operatorname{div} \chi^{u(\sigma)} + D\Big|_{U_\sigma} \geq 0 \right)$$

Lemma: Σ is simplicial $\Rightarrow$ Every Weil divisor D is $\mathbb{Q}$- Cartier.

§ Line Bundles, Picard group.

$\mathrm{Pic}(X) =$ group of line bundles / isom

$=$ Cartier divisors / {principle Cartier divisor}

$A_{n-1}(X) =$ Weil divisors / $\{[\mathrm{div}(f)]\}$

X is normal : $\mathrm{Pic}(X) \hookrightarrow A_{n-1}(X)$

$X =$ toric, $u \in M$.

$\mathrm{div} : M \to \{\mathrm{Div}_T X = T\text{-Cartier divisors}\}$

Compute $\mathrm{Pic}(X)$, $A_{n-1}(X)$ with T-Cartier (Weil) divisors :

Prop: $X = X(\Sigma)$. Σ fan not contained in any proper subspace of $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathrm{Div}_T X \arrow[r] \arrow[d, hook] & \mathrm{Pic}(X) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M \arrow[r] & \bigoplus_{i=1}^{d} \mathbb{Z}\cdot D_i \arrow[r] & A_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

$$m \mapsto \{\langle m, v_i\rangle\}_i$$

$$\{a_i\}_i \mapsto \sum_i a_i D_i$$

$d = |\Sigma(1)|$, rank $\mathrm{Pic}(X) \leq$ rank $A_{n-1}(X) = d-n$.

$\mathrm{Pic}(X)$ = subgp of $\bigoplus M(\sigma) \simeq \mathbb{Z}^{|\Sigma(n)|}$ is abelian.

proof: $X \setminus \bigcup D_i = T_N$ is affine.

so all Cartier, Weil divisors on T_N are principal

Coro: If all maximal cones of Σ are n-dim'l, then $\mathrm{Pic}(X(\Sigma)) \simeq H^2(X(\Sigma), \mathbb{Z})$.

pf: $\mathrm{Div}_T(X) = \mathrm{Ker}\left(\bigoplus_i M/M(\sigma_i) \simeq \bigoplus_i M \longrightarrow \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\right)$

$H^2(X; \mathbb{Z}) = \mathrm{Ker}\left(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \longrightarrow \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)\right)$

$$\begin{array}{ccc} \mathrm{Div}_T(X) & \twoheadrightarrow & H^2(X; \mathbb{Z}) \\ \bigoplus u_i & \longmapsto & \bigoplus (u_j - u_i) \end{array}$$

Surj: $u_{ij} + u_{jk} = u_{ik}$. $u_{ij} = -u_i + u_j$ ✓, $u_{jk} = -u_j + u_k$ ✓,

$\rightsquigarrow$ $u_{ik} = -u_i + u_k$. ✓.

has kernel M. (because $u_i \equiv u_j$).

$\therefore$ $\mathrm{Pic}(X(\Sigma)) \simeq H^2(X(\Sigma); \mathbb{Z})$.

Ex: $\mathrm{Pic}(X(\Delta)) \to H^2(X; \mathbb{Z})$ may be not surjective.

$X = T_N$: algebraic bundle $\mathrm{Pic}(X) = 0$.

$(n=2)$ $\qquad H^2(X;\mathbb{Z}) = \mathbb{Z}$

The torus has analytic line bundles that are not algebraic.

Exercise. Let Δ be a fan such that all of its maximal cones are n-dimensional. Show that the following are equivalent:

(i) Δ is simplicial;

(ii) Every Weil divisor on $X(\Delta)$ is a $\mathbb{Q}$-Cartier divisor;

(iii) $\mathrm{Pic}(X(\Delta))\otimes\mathbb{Q} \to A_{n-1}(X(\Delta))\otimes\mathbb{Q}$ is an isomorphism;

(iv) $\mathrm{rank}(\mathrm{Pic}(X(\Delta))) = d - n.$ (11)

Cartier divisor $D \longleftrightarrow \left\{ u(\sigma) \in M/M(\sigma) \;\middle|\; \ker: \bigoplus_i \frac{M}{M(\sigma_i)} \to \bigoplus_{i<j} \frac{M}{M(\sigma_i\cap\sigma_j)} \right\}$

$\Rightarrow \quad \psi_D(v) := \langle u(\sigma), v\rangle \quad v \in \sigma.$

Well-define: if $v \in \sigma\cap\tau$, then $u(\sigma) = u(\tau)$

in $M/M(\sigma\cap\tau)$

so $\langle u(\sigma), v\rangle = \langle u(\tau), v\rangle.$

$$\left\{ \begin{array}{l} f \text{ is continuous} \\ \text{piecewise linear \& integral} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} T\text{-Cartier divisors} \\ D \end{array} \right\}$$

$$\psi_D \longleftarrow D$$

$D = \sum a_j D_j$, then v_i

$\psi_D(v_i) = \langle u(\sigma), v_i\rangle = -a_i$ (as given in Ima.).

Prop: $\cdot\ \psi_{D+E} = \psi_D + \psi_E.$

$\cdot\ \psi_{mD} = m\psi_D.$

- $\psi_{\mathrm{div}(\chi^u)}(\cdot) = \langle -u, \cdot \rangle$

- If $D \sim E$, then $\exists\, u \in M$ s.t.

  $\psi_D - \psi_E = \langle u, \cdot \rangle$

$P_D :=$ rational convex polyhedron in $M_{\mathbb{R}}$ defined by

$$= \{ u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -a_i \quad \forall i \}$$

$$= \{ u \in M_{\mathbb{R}} \mid u \geq \psi_D \text{ on } |\Delta| \}.$$

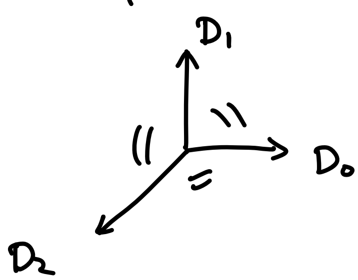
lemma: The global sections of $\mathcal{O}(D)$ are

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \chi^u.$$

proof: $\Gamma(U_\sigma, \mathcal{O}(D)) = \bigoplus_{u \in P_D(\sigma)} \mathbb{C} \cdot \chi^u$

$$P_D(\sigma) = \{ u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -a_i \quad \forall v_i \in \sigma \}.$$

Example: $\mathbb{P}^2$

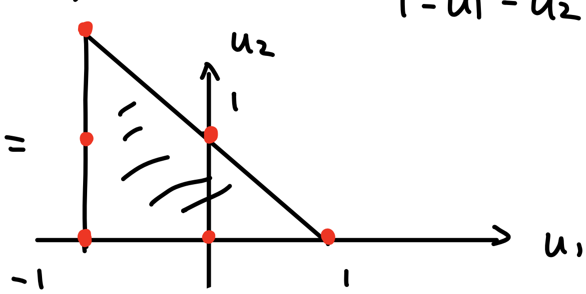


$D := D_0 + D_2$.

$v_0 = (1,0)$, $v_1 = (0,1)$, $v_2 = (-1,-1)$

$$P_D = \left\{ u = (u_1, u_2) \;\middle|\; \begin{array}{l} u_1 \geq -1, \quad u_2 \geq 0, \\ -u_1 - u_2 \geq -1 \end{array} \right\}.$$

=



$\Gamma(X, \mathcal{O}(D_0 + D_2)) = \mathbb{C}\chi_1^{-1}\chi_2^{2} \oplus \mathbb{C}\chi_1^{-1}\chi_2 \oplus \mathbb{C}\chi_2 \oplus \mathbb{C}\chi_1^{-1} \oplus \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot \chi_1$

$h^0(X, \mathcal{O}(2)) = \binom{2+2}{2} = \binom{4}{2} = 6.$

Rmk:

- $P_{mD} = m P_D$
- $P_{D+\operatorname{div}(\chi^u)} = P_D - u$
- $P_D + P_E \subset P_{D+E}$.

- Base point free & Ampleness criterion.

Prop: Assume $|\Sigma| = N_{\mathbb{R}}$ i.e. $X(\Sigma)$ is complete.

Let D be a T-Cartier divisor on $X(\Sigma)$. Then $\mathcal{O}(D)$ is

(1) base point free $\iff$ ψ_D is upper convex

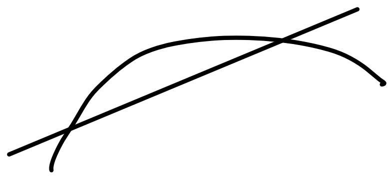
$\iff$ $\langle u(\sigma), v_\rho \rangle \geq -a_\rho$ whenever $\rho \not\subset \sigma$.

(2) ample $\iff$ ψ_D is strictly upper convex

$\iff$ $\langle u(\sigma), v_\rho \rangle > -a_\rho$ whenever $\rho \not\subset \sigma$

(3) very ample $\iff$ ψ_D is strictly upper convex and every n-dim'l cone σ, S_σ is generated by

$\{u - u(\sigma) : u \in P_D \cap M\}$.



upper convex

$$\psi(t\cdot v+(1-t)w) \geq t\psi(v)+(1-t)\psi(w).$$

Prop: If X is complete and nonsingular, then T-Cartier divisor D is ample $\Leftrightarrow$ very ample.

Rmk: $\exists$ X complete, nonsingular, but not projective.

Prop: (1) If $|\Sigma|$ is convex, and $\mathcal{O}(D)$ is base point free, then $H^p(X,D)=0$ for $p>0$.

(2) If X is complete, $\mathcal{O}(D)$ basepoint free, then $\chi(X,D) = \dim H^0(X,D) = \#(P_D \cap M)$.

$X = X(\Sigma)$ simplicial and projective $\Rightarrow$ imply complete/proper/k.

- $Cpl(\Sigma) = \{ a = \sum_{i=1}^{r} a_i D_i \in A^{+}_{n-1}(X)\otimes\mathbb{R} \mid \psi_a \text{ is convex} \}$

  $= \mathrm{Nef}(X_\Sigma)$

  Kähler cone(X) = Ample cone (X_Σ).

  = interior of $cpl(\Sigma)$.

- $P_D \longleftrightarrow$ polytope $\Delta_p \subset \Sigma$, $X = X(\Sigma)$.

D ample on X

§ Intersection Theory. Chow ring. [Cox]. Ch 12.

§ The Cohomology Ring.

X_Σ = complete, simplicial.

$H^\bullet(X_\Sigma, \mathbb{Q}) = \bigoplus_{k=0}^{2n} H^k(X_\Sigma, \mathbb{Q})$ $\quad n = \dim X_\Sigma$,

$H^\bullet_{T_N}(X_\Sigma, \mathbb{Q})$

$\rho_1, \cdots, \rho_r$ rays of $\Sigma(1)$.

$\mathbb{Q}[x_1, \cdots, x_r]$

I ideal $= \langle x_{i_1} \cdots x_{i_s} \mid i_j$ distinct, $\rho_{i_1} + \cdots + \rho_{i_s}$ not a cone of $\Sigma \rangle$

$:=$ Stanley-Reisner ideal.

$\mathcal{J} = \left\langle \sum_{i=1}^{r} \langle m, v_i \rangle x_i \;\middle|\; m \text{ ranges over } M \right\rangle.$

$R_\mathbb{Q}(\Sigma) := \mathbb{Q}[x_1, \cdots, x_r] / (\mathcal{I} + \mathcal{J})$

$$\begin{aligned} R_\mathbb{Q}(\Sigma) &\longrightarrow H^\bullet(X_\Sigma, \mathbb{Q}) \\ x_i &\longmapsto [D_i]. \end{aligned}$$ ring hom.

Theorem 12.4.1. *Let Σ be complete and simplicial. Then the map* (12.4.4) *is an isomorphism:*

$$R_\mathbb{Q}(\Sigma) \simeq H^\bullet(X_\Sigma, \mathbb{Q}).$$

Thus, in even degrees, $H^{2k}(X_\Sigma, \mathbb{Q})$ is isomorphic to $R_\mathbb{Q}(\Sigma)_k$, and in odd degrees, $H^{2k+1}(X_\Sigma, \mathbb{Q})$ is zero.

Examples:

1) $\mathbb{P}^n$: $v_i = e_i, \quad i = 1, \cdots, n,$

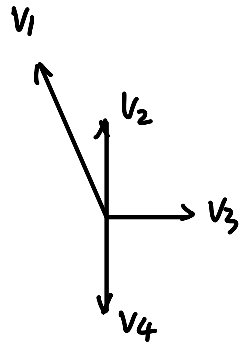
$v_0 = -e_1 - \cdots - e_n.$

$I = \langle x_0 \cdots x_n \rangle$

$J = \langle x_1 - x_0, \cdots, x_n - x_0 \rangle.$

$$H^{\cdot}(\mathbb{P}^n, \mathbb{Q}) \simeq \mathbb{Q}[x_0, \cdots, x_n] \Big/ \langle x_0 \cdots x_n,\ x_1 - x_0, \cdots, x_n - x_0 \rangle$$

$$\simeq \mathbb{Q}[x_0] \Big/ \langle x_0^{n+1} \rangle.$$



2). $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(r)) = \mathcal{H}_r$

$v_1 = -e_1 + r e_2, \quad v_2 = e_2, \quad v_3 = e_3, \quad v_4 = -e_2$

$I = \langle x_1 x_3,\ x_2 x_4 \rangle, \qquad J = \langle -x_1 + x_3,\ \ r x_1 + x_2 - x_4 \rangle$

$$H^{\cdot}(\mathcal{H}_r, \mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, x_2, x_3, x_4]}{\langle x_1 x_3, x_2 x_4, -x_1 + x_3, r x_1 + x_2 - x_4 \rangle}$$

$$\cong \frac{\mathbb{Q}[x_1, x_2]}{\langle x_1^2, x_2^2 + r x_1 x_2 \rangle}$$

$$\begin{pmatrix} D_1 \cdot D_1 & D_1 \cdot D_2 \\ D_2 \cdot D_1 & D_2 \cdot D_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix}.$$

Theorem 12.4.4 (Jurkiewicz-Danilov). *Let X_Σ be a smooth complete toric variety. For the polynomial ring $\mathbb{Z}[x_1,\ldots,x_r]$ with variables indexed by $\rho_1,\ldots,\rho_r \in \Sigma(1)$, let $\mathscr{I}$ and $\mathcal{J}$ be the ideals in $\mathbb{Z}[x_1,\ldots,x_r]$ generated by the polynomials in* (12.4.2) *and* (12.4.3)*, and define*

$$R(\Sigma) = \mathbb{Z}[x_1,\ldots,x_r]/(\mathscr{I}+\mathcal{J}).$$

Then $x_i \mapsto [D_{\rho_i}]$ induces a ring isomorphism $R(\Sigma) \simeq H^\bullet(X_\Sigma,\mathbb{Z})$. □

Proof: Equivariant cohomology:

$$\Lambda_G = H^*_G(pt, \mathbb{Z})$$

$$\Lambda_T = H^*_T(pt, \mathbb{Z}) = \mathbb{Z}[x_1, \cdots, x_n]$$

Page 605 [Cox]

Theorem 12.3.12. *Let Σ be a complete simplicial fan in $N_\mathbb{R} \simeq \mathbb{R}^n$. Then the Betti numbers of X_Σ are given by*

$$b_{2k}(X_\Sigma) = \sum_{i=k}^{n} (-1)^{i-k}\binom{i}{k}|\Sigma(n-i)|$$

$b_{2k}(X_\Sigma) := \dim H^{2k}(X_\Sigma, \mathbb{Q})$

and satisfy

$$b_{2k}(X_\Sigma) = b_{2n-2k}(X_\Sigma).$$

§ Chow group / ring.

$$A_k(X) = Z_k(X) / \mathrm{Rat}_k(X)$$

$$A^k(X) = A_{n-k}(X)$$

$$A^k(X) \times A^l(X) \longrightarrow A^{k+l}(X)$$

$$A^\bullet(X) = \bigoplus_{k=0}^{n} A^k(X)$$ Chow ring.

$$A^\bullet(X) \longrightarrow H^\bullet(X, \mathbb{Z})$$ ring hom.

Toric case: If X_Σ is a complete simplicial toric variety of dim n, then intersection product can be defined on rational cycles,

$$A^\bullet(X_\Sigma)_\mathbb{Q} = A^\bullet(X_\Sigma) \otimes_\mathbb{Z} \mathbb{Q} = \bigoplus_{k=0}^{n} A^k(X_\Sigma) \otimes_\mathbb{Z} \mathbb{Q}$$

Lemma: $[V(\sigma)]$, $\sigma \in \Sigma$ generate $A^\bullet(X_\Sigma)$ as an abelian group.

lemma: Assume X_Σ is complete and simplicial. If $\rho_1, \cdots, \rho_d \in \Sigma(1)$ are distinct, then in $A^\bullet(X_\Sigma)_\mathbb{Q}$ we have

$$[D_{\rho_1}][D_{\rho_2}] \cdots [D_{\rho_d}] = \begin{cases} \frac{1}{\mathrm{mult}(\sigma)}[V(\sigma)] & \text{if } \sigma = \rho_1 + \cdots + \rho_d \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{mult}(\sigma) = [\mathbb{Z}\sigma : \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_d].$$

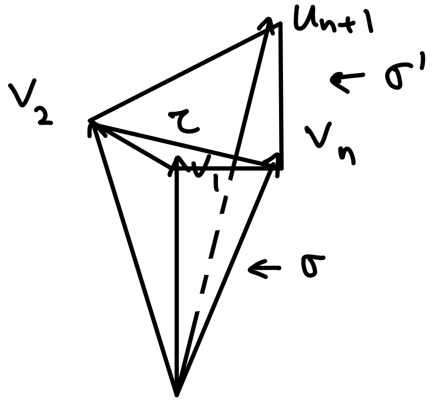
lemma: Let $\rho \in \Sigma(1)$, $\sigma \in \Sigma$ not containing ρ,

$$[D_\rho] \cdot [V(\sigma)] = \begin{cases} \frac{\mathrm{mult}(\sigma)}{\mathrm{mult}(\tau)}[V(\tau)] & \text{if } \tau = \rho + \sigma \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

lma: $\tau \in \Sigma(n-1)$, $\sigma = \mathrm{Cone}(v_1, \cdots, v_n)$

$\sigma' = \mathrm{Cone}(v_2, \cdots, v_{n+1})$

$\tau = \sigma \cap \sigma' = \mathrm{Cone}(v_2, \cdots, v_n)$



$$\alpha v_1 + \sum_{i=2}^{n} b_i v_i + \beta v_{n+1} = 0.$$

Proposition 6.4.4. *The relations* (6.4.4) *and* (6.4.5) *are equal after multiplication by a positive constant. Furthermore:*

(a) $D_\rho \cdot V(\tau) = 0$ *for all* $\rho \notin \{\rho_1, \ldots, \rho_{n+1}\}$.

(b) $D_{\rho_1} \cdot V(\tau) = \dfrac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)}$ *and* $D_{\rho_{n+1}} \cdot V(\tau) = \dfrac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma')}$.

• (c) $D_{\rho_i} \cdot V(\tau) = \dfrac{b_i\,\mathrm{mult}(\tau)}{\alpha\,\mathrm{mult}(\sigma)} = \dfrac{b_i\,\mathrm{mult}(\tau)}{\beta\,\mathrm{mult}(\sigma')}$ *for* $i = 2, \ldots, n$.

e.g. $\mathcal{H}_r$: $v_1 + (-r v_2) + v_3 = 0$

$$\therefore \quad D_2 \cdot D_2 = D_2 \cdot V(\tau) = -r.$$

· Lemma : Σ simplicial

$$A^p(X) \times A^q(X) \longrightarrow A^{p+q}(X)$$

$$\underset{\substack{\uparrow \\ \text{cone dim } p}}{V(\sigma)} \cdot V(\tau) = \frac{\mathrm{mult}(\sigma) \cdot \mathrm{mult}(\tau)}{\mathrm{mult}(\gamma)} V(\gamma)$$

γ = cone of $p+q$ spanned by σ & τ and $\dim \gamma = p+q$

The Chow Ring of a Toric Variety. As in §12.4, write $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$. This gives the ring

$$R_{\mathbb{Q}}(\Sigma) = \mathbb{Q}[x_1, \ldots, x_r]/(\mathscr{I} + \mathscr{J})$$

for $\mathscr{I}$ and $\mathscr{J}$ as in (12.4.2) and (12.4.3). Then Lemma 12.5.2 and (12.5.4) imply that $[x_i] \mapsto [D_{\rho_i}] \in A^1(X_\Sigma)_{\mathbb{Q}}$ defines a ring homomorphism

$$R_{\mathbb{Q}}(\Sigma) \longrightarrow A^\bullet(X_\Sigma)_{\mathbb{Q}}. \tag{12.5.8}$$

We also have the ring homomorphism $A^\bullet(X_\Sigma)_{\mathbb{Q}} \to H^\bullet(X_\Sigma, \mathbb{Q})$ from (12.5.2).

Theorem 12.5.3. *If X_Σ is complete and simplicial, then*

$$R_{\mathbb{Q}}(\Sigma) \simeq A^\bullet(X_\Sigma)_{\mathbb{Q}} \simeq H^\bullet(X_\Sigma, \mathbb{Q}),$$

where the maps are given by (12.5.8) *and* (12.5.2).

§ Characteristic Class, HRR.

Prop : X nonsingular toric, $D_1, \cdots, D_d$ irred T-divisors,

then (1) $K_X = -\sum D_i$, $\Omega_X^n = \mathcal{O}_X(-\sum_{i=1}^d D_i)$.

(2) $0 \to \Omega_X^1 \to \Omega_X^1(\log D) \xrightarrow{\text{res}} \bigoplus_{i=1}^d \mathcal{O}_{D_i} \to 0$.

$$\sum_i f \frac{dX_i}{X_i} \mapsto f|_{D_i} \qquad D = \sum_{i=1}^d D_i$$

(3) $\Omega_X^1(\log D)$ is trivial.

$$M \otimes_{\mathbb{Z}} \mathcal{O}_X \xrightarrow{\sim} \Omega_X^1(\log D)$$
$$u \otimes 1 \mapsto \frac{d(\chi^u)}{\chi^u}$$

(2)(3) $\Rightarrow$ (1).

(4) generalized Euler Seq : X smooth, complete.

$$0 \to \Omega_{X_\Sigma}^1 \to \bigoplus_{i=1}^d \mathcal{O}_{X_\Sigma}(-D_i) \to \mathrm{Pic}(X_\Sigma) \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma} \to 0.$$

(4) $\Rightarrow$ (1) too.

(4): $0 \to M \to \bigoplus_{i=1}^{d} \mathbb{Z}\cdot D_i \to Cl(X) \to 0$

$$\xrightarrow[\text{exact}]{\otimes \mathcal{O}_X \text{ free sheaf}} \quad 0 \to M\otimes_{\mathbb{Z}} \mathcal{O}_X \to \bigoplus_{i=1}^{d} \mathcal{O}_X \to Cl(X)\otimes \mathcal{O}_X \to 0$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^1_X \arrow[r] \arrow[d] & \bigoplus_{i=1}^{d} \mathcal{O}_X(-D_i) \arrow[r] \arrow[d] & Cl(X)\otimes \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M\otimes_{\mathbb{Z}} \mathcal{O}_X \arrow[r] \arrow[d] & \bigoplus_{i=1}^{d} \mathcal{O}_X \arrow[r] \arrow[d] & Cl(X)\otimes \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus \mathcal{O}_{D_i} \arrow[r] \arrow[d] & \bigoplus \mathcal{O}_{D_i} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

snake lemma.

exact　　exact　　exact

$$\begin{array}{ccc}
u\otimes f & \longmapsto & (\langle u, v_i\rangle f)_i \\
\downarrow & & \downarrow \\
\left(\operatorname{res}_{D_i}\left(\frac{d\chi^u}{\chi^u}\cdot f\right)\right)_i = \langle u, v_i\rangle f|_{D_i} & \longmapsto & (\langle u, v_i\rangle f|_{D_i})_i
\end{array}$$

Chern class: $X = X_\Sigma$ smooth complete.

(1) $c(T_X) = \prod_\rho (1+[D_\rho]) = \sum_{\sigma\in\Sigma} [V(\sigma)]$

(2) $c_1 = [\sum_\rho D_\rho] = [-K_X]$

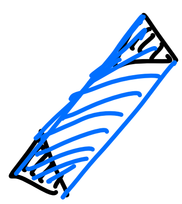
(3) $$Td(X) = \prod_{\rho \in \Sigma(1)} \frac{[D_\rho]}{1 - e^{-[D_\rho]}} \in H^{\cdot}(X, \mathbb{Q})$$

§ Polytopes & Homogeneous Coordinates.

A polytope $\Delta \subset M_{\mathbb{R}}$ is a convex hull of a finite set of points. $\dim \Delta$ = dim of subspace spanned by $\{m_1 - m_2 : m_1, m_2 \in \Delta\}$. Δ is integral if $\mathrm{Vertex}(\Delta) \subseteq M$. Facet of Δ = codim 1 face of Δ.

$\Delta_1, \cdots, \Delta_k$ in $M_{\mathbb{R}}$, the convex hull of $\Delta_1, \cdots, \Delta_k$

$$\mathrm{Conv}(\Delta_1, \cdots, \Delta_k)$$



The Minkowski sum is

$$\Delta_1 + \cdots + \Delta_k = \{m_1 + \cdots + m_k \mid m_i \in \Delta_i\}.$$

$$k\Delta := \underbrace{\Delta + \cdots + \Delta}_{k \text{ times}}$$

§ Polytopes . Toric Varieties.

$t_0^k \chi^m$ monomials, $m \in k\Delta$.

$t_0^k \chi^m + t_0^l \chi^{m'} = t_0^{k+l} \chi^{m+m'}$, $m \in k\Delta$, $m' \in l\Delta$

$\mathbb{C}$-alg $S_\Delta := \mathbb{C}[t_0^k \chi^m \mid k, m]$.

$\deg t_0^k \chi^m := k$.

Let $\mathbb{P} = \mathbb{P}_\Delta = \mathrm{Proj}(S_\Delta)$.

What's the fan of $\mathbb{P}_\Delta$?

$\underset{\text{face}}{F} \subset \Delta$,

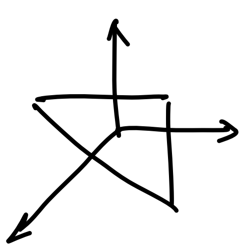
$\sigma_F^\vee := \{\lambda(m - m') \mid m \in \Delta, m' \in F, \lambda \geq 0\} \subseteq M_{\mathbb{R}}$

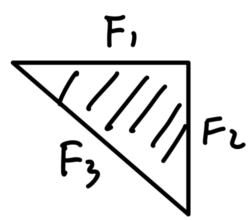
σ_F = dual cone of $\sigma_F^\vee \subseteq N_{\mathbb{R}}$.

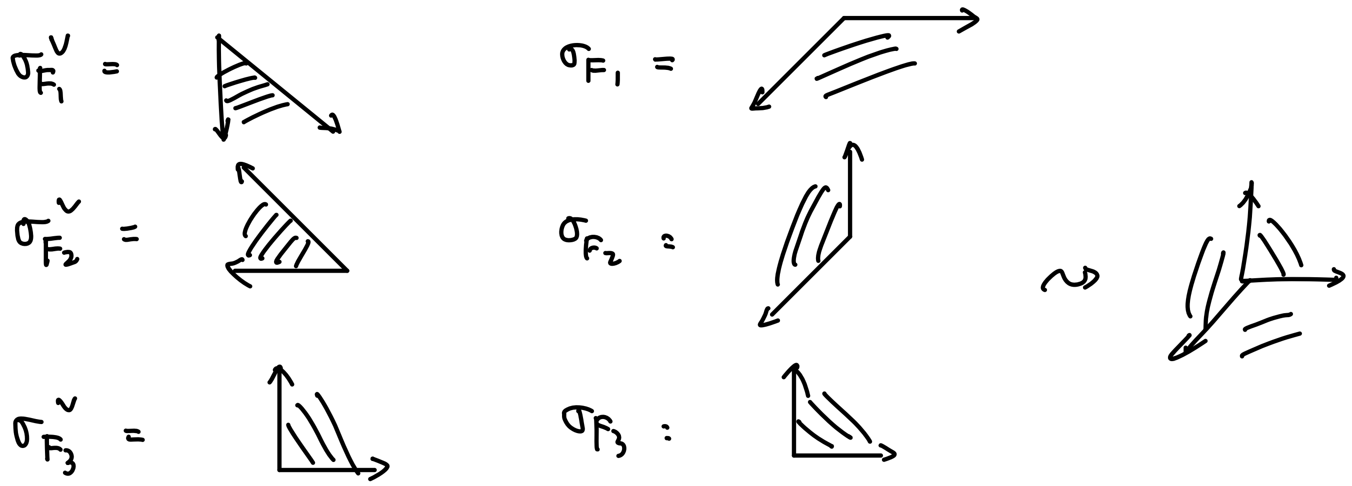
$\Sigma = \{\sigma_F\}_F$ = normal fan of Δ

Σ: a complete fan $\qquad \mathbb{P}_\Delta = X(\Sigma)$.

Note: Δ_P = normal fan of $\underline{P \text{ polytope}}$ in [Fulton]

e.g. 

Let Δ = 



$\Delta^\circ = \{ v \in N_{\mathbb{R}} : \langle m, v \rangle \geq -1$ for all $m \in \Delta \} \subset N_{\mathbb{R}}$.

Lemma: Σ obtained from cones over the proper faces of Δ° is the normal fan of Δ.

Definition 3.5.3. *A n-dimensional integral polytope $\Delta \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ is* reflexive *if the following two conditions hold:*

(*i*) *All facets Γ of Δ are supported by an affine hyperplane of the form $\{m \in M_{\mathbb{R}} : \langle m, v_\Gamma \rangle = -1\}$ for some $v_\Gamma \in N$.*

(*ii*) $\mathrm{Int}(\Delta) \cap M = \{0\}$.

Reflexive polytopes have a very pretty combinatorial duality. Let Δ be an integral polytope, and let Δ° be the *polar* polytope defined in Section 3.2.1. Besides $(\Delta^\circ)^\circ = \Delta$, [**Batyrev4**] shows that the basic duality between Δ and Δ° is as follows.

Lemma 3.5.4. *Δ is reflexive if and only if Δ° is reflexive.*

Reflexive polytopes are interesting in this context because of the following result, which characterizes when $\mathbb{P}_\Delta$ is Fano.

Proposition 3.5.5. *Δ is reflexive if and only if $\mathbb{P}_\Delta$ is Fano.*

Lemma 3.5.6. *Let $X = \mathbb{P}(q_0, \ldots, q_n)$ be a weighted projective space, and let $q = \sum_{i=0}^n q_i$. Then X is Fano if and only if $q_i | q$ for all i.*

Homogeneous coordinates.

$S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$. $\quad r = |\Sigma(1)|$

$x^D = \prod_\rho x_\rho^{a_\rho} \quad D = \sum_\rho a_\rho D_\rho$ effective T-Weil divisor.

$\deg x^D := [D] \in A_{n-1}(X)$.

$S =$ homogeneous coordinate ring of X.

$$0 \to M \to \bigoplus_{i=1}^{|\Sigma(1)|} \mathbb{Z}\cdot D_i \to A_{n-1}(X) \to 0.$$

$\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$: $\quad G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*)$

$$1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \xrightarrow{\phi} T_N \to 1$$

$g \in G, \quad a = (a_\rho) \in \mathbb{C}^{\Sigma(1)} = \mathrm{Spec}(S)$

$$g \cdot a = (g[D_\rho]a_\rho).$$

$$\phi(f)(m) = \prod_{i=1}^r f(v_i)^{\langle m, v_i\rangle}$$

$$G = \{(t_1, \dots, t_r) \in (\mathbb{C}^*)^r \mid \textstyle\prod_{i=1}^r t_i^{\langle m, v_i\rangle} = 1 \quad \forall m \in M\}.$$

Thm: $|\Sigma| = N_{\mathbb{R}}$, then

(i) X is the categorical quotient of $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ by G

(ii) X is the geometrical quotient of $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ by G

iff X is simplicial.

$$X = \left(\mathbb{C}^{\Sigma(1)} - Z(\Sigma)\right)/G.$$

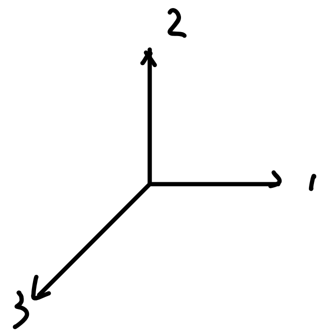
$$Z(\Sigma) = \bigcup_S V(S) = \bigcup_S \{x_\rho = 0 \mid \rho \in S\}$$

$$S \subseteq \Sigma(1).$$

$\uparrow$

不是 σ 的边，但任一 proper subset 是 τ 的某个 component.

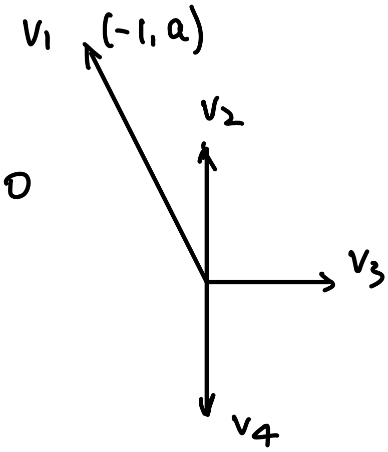
e.g. $\mathbb{P}^2 = \left(\mathbb{C}^3 - \{x_0 = x_1 = x_2 = 0\}\right)/\mathbb{C}^*$



$$\phi : (\mathbb{C}^\times)^3 \longrightarrow (\mathbb{C}^\times)^2$$

$$(t_1, t_2, t_3) \longmapsto (t_1 t_3^{-1}, t_2 t_3^{-1})$$

$$G = \{(t, t, t) \mid t \in \mathbb{C}^*\}$$



e.g. $0 \to M = \mathbb{Z}^2 \xrightarrow{B} \bigoplus_{i=1}^{4} \mathbb{Z} \cdot D_i \xrightarrow{A} \mathbb{Z}^2 \to 0$

$$B = \begin{pmatrix} -1 & a \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & a & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$1 \to G \xrightarrow{\begin{pmatrix} 1 & 0 \\ a & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} (\mathbb{C}^*)^4 \xrightarrow{\begin{pmatrix} -1 & 0 & 1 & 0 \\ a & 1 & 0 & -1 \end{pmatrix}} T_N = (\mathbb{C}^*)^2 \to 1$$

group $G = \{(t, t^{-a}u, t, u) \mid t, u \in \mathbb{C}^*\}$.

§ Moment Map

P convex polytope in $M_{\mathbb{R}}$ with vertices in M

$\rightsquigarrow X(\Delta_P)$,

Moment map:

$$\mu : X(\Delta_P) \longrightarrow M_{\mathbb{R}} \qquad \rightsquigarrow \quad X_{\geq} = X/S_N \xrightarrow[\sim]{\text{homeo}} P$$

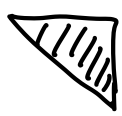
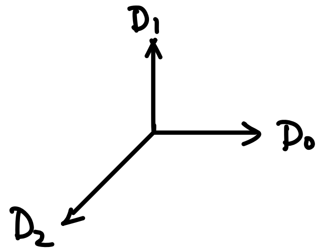
$$\mu(x) = \frac{1}{\sum_{u \in P\cap M} |\chi^u(x)|} \sum_{u \in P \cap M} |\chi^u(x)|\, u$$

$$\text{convex polytope } P \xrightarrow{\text{normal cone}} \Sigma = \Delta_P \longrightarrow X(\Sigma)$$

$$P \xleftarrow[\text{moment map.}]{} X(\Sigma) \quad \text{complete, nonsingular}$$

$=$ ample D, P_D $\qquad$ $P_D = P \longleftrightarrow$ ample divisor D

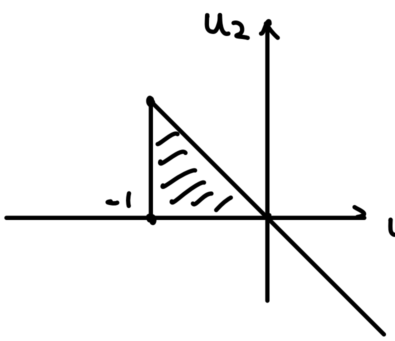
Example:

 $\longrightarrow$ 

$\longrightarrow$ $\mathbb{P}^2$

$D = D_0$

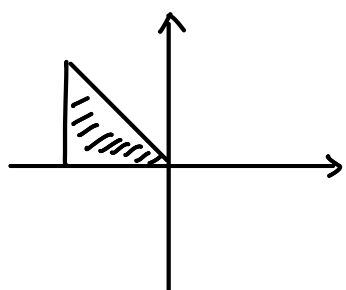
$$P_D = \left\{ (u_1, u_2) \;\middle|\; \begin{array}{l} u_1 \geq -1,\ u_2 \geq 0 \\ -u_1 - u_2 \geq 0 \end{array} \right\}$$



Compute: $x = (t_1, t_2)$

$u = (-1,0),\ (-1,1).\ (0,0)$

$$\mu(t_1, t_2) = \frac{|t_1^{-1}|(-1,0) + |t_1^{-1}t_2|(-1,1) + 1(0,0)}{|t_1^{-1}| + |t_1^{-1}t_2| + 1}$$



(Delzant, 1990) (toric mfd)/$\sim$ $\xleftrightarrow{1:1}$ {Delzant polytopes}

$$(\mathbb{P}_\Delta, \omega_\Delta, \mathbb{T}^n, \mu) \longleftrightarrow \mu(\mathbb{P}_\Delta)$$

$\parallel$

§ Moment map / $\mathbb{C}^r$ $\leftrightarrow$ Symplectic Reduction.

Hamiltonian action: $G \circlearrowright \mathrm{Sympl}(M, \omega)$ is Hamiltonian

if $\exists\, \mu : M \to \mathfrak{g}^*$ (moment map) satisfying

(1) $\forall X \in \mathfrak{g}$, $\mu^X := \langle \mu, X \rangle : M \to \mathbb{R}$
$$p \mapsto \langle \mu(p), X \rangle$$

s.t. $d\mu^X = \omega(X^\#, -)$

where $X^\#$ is generated by $\{ \exp tX(e) \mid t \in \mathbb{R} \}$.

(2) $\mu(g \cdot p) = \mathrm{Ad}^*_g \circ \mu(p) \quad \forall\, p \in M$

e.g.

- $M = \mathbb{C}^r$, $\omega_{\mathbb{C}^r} = \sum_{i=1}^r dx_i \wedge dy_i \qquad (z_i = x_i + \sqrt{-1}\, y_i)$

$G = U(1)^r$

$\mathfrak{g} = \{ \lambda \mid (\lambda_1, \cdots, \lambda_r) \in \mathbb{R}^r, |\lambda_i| = 1 \}$.

$$\mathfrak{g} \longrightarrow U(1)^r \longrightarrow \mathrm{Aut}(\mathbb{C}^r)$$
$$\lambda \mapsto \exp(i\lambda) \longmapsto \left(v \mapsto \exp(i\lambda) \cdot v \right).$$
$$= (\exp(i\lambda_1), \cdots, \exp(i\lambda_r)).$$

$$\lambda \longmapsto \{ \text{a flow on } \mathbb{C}^r : v \mapsto \exp(i\lambda t) \cdot v \}.$$
with $X_\lambda = \sum_{i=1}^r \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \right)$

$\mu : \mathbb{C}^r \longrightarrow \mathfrak{g}^* = \mathbb{R}^r$

$\mu(z_1, \cdots, z_r) = \frac{1}{2}\left(|z_1|^2, \cdots, |z_r|^2 \right)$.

Toric varieties $\leftrightsquigarrow$ Symplectic reduction.

$$G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*),$$

maximal compact subgp: $G_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), U(1))$

Lie alg $\quad \mathfrak{g}_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{R})$

$$\mathfrak{g}^*_{\mathbb{R}} = A_{n-1}(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

$r = |\Sigma(1)|$, $\quad G \subseteq (\mathbb{C}^*)^r \rightsquigarrow G_{\mathbb{R}} \subseteq U(1)^r \circlearrowright \mathbb{C}^r$.

$\rightsquigarrow$

$$\mu_\Sigma : \mathbb{C}^r \xrightarrow{\mu} (\mathbb{R}^r)^* \xrightarrow{p} \mathfrak{g}_{\mathbb{R}}^* = A_{n-1}(X) \otimes \mathbb{R} \cong \mathbb{R}^{r-n}.$$

p is from the exact seq:

$$0 \to M \to \bigoplus \mathbb{Z}\cdot D_i \to A_{n-1}(X) \to 0.$$

$\otimes_{\mathbb{Z}} \mathbb{R}$:

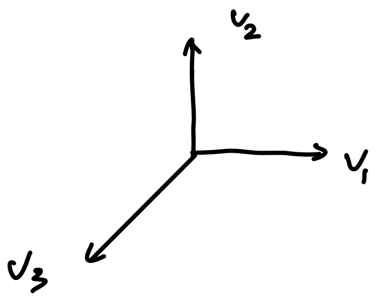
$$0 \to M_{\mathbb{R}} \to (\mathbb{R}^r)^* \xrightarrow{p} \mathfrak{g}_{\mathbb{R}}^* \to 0$$

Thm: If $X = X_\Sigma$ is projective and simplicial, and $a \in A_{n-1}(X) \otimes \mathbb{R} \cong H^{1,1}(X, \mathbb{R})$ is Kähler (Ample), then $\mu_\Sigma^{-1}(a) \subset \mathbb{C}^r - Z(\Sigma)$ and

$$\mu_\Sigma^{-1}(a)/G_{\mathbb{R}} \to (\mathbb{C}^r - Z(\Sigma))/G = X$$

is an orbifold diffeo. Furthermore, the symplectic form ω on $\mathbb{C}^r$, $\quad \omega|_{\mu_\Sigma^{-1}(a)}$ descends to a symplectic form on $\mu_\Sigma^{-1}(a)/G_{\mathbb{R}}$, whose cohomology class is identified with $a \in H^2(X, \mathbb{R})$ via the above diffeo.

e.g. $\mathbb{P}^2$:

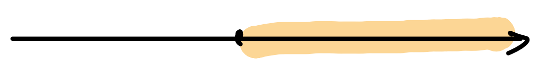


$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{(1\ 1\ 1)} A_1(X) \longrightarrow 0$$

$$0 \longrightarrow G \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

$\therefore \mu_\Sigma(z_1, z_2, z_3) = \frac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2)$

Kähler cone :



$[D_1] = [D_2] = [D_3]$

$a > 0$

$\therefore \mu_\Sigma^{-1}(1) = \mathbb{C}^3 - \{0\}$.

$G \cong \mathbb{C}^*$ $\qquad t : (z_1, z_2, z_3) \mapsto (tz_1, tz_2, tz_3)$.

$\therefore \mu_\Sigma^{-1}(1)/G = \mathbb{C}^3 - \{0\}/\mathbb{C}^* = \mathbb{P}^2$.

Gromov-Witten Invariant; Toric complete intersection.

We first set up some notation. For each $\rho \in \Sigma(1)$, we abuse notation and let D_ρ also denote the cohomology class of the associated divisor D_ρ in $H^2(X_\Sigma)$. Following [**Givental4**], we put $\mathcal{L}_i(\beta) = \int_\beta c_1(\mathcal{L}_i)$ and $D_\rho(\beta) = \int_\beta D_\rho$. We also pick an integral basis $T_1, \dots, T_r$ of $H^2(X_\Sigma, \mathbb{Z})$ which lie in the closure of the Kähler cone. As usual, we set $\delta = \sum_{i=1}^r t_i T_i$.

We now define two cohomology-valued formal functions. We begin with $I_\mathcal{V}$, which is given by

$$I_\mathcal{V} = e^{(t_0+\delta)/\hbar}\,\mathrm{Euler}(\mathcal{V}) \times$$
$$\sum_{\beta \in M(X_\Sigma)} q^\beta \frac{\prod_{i=1}^{\ell} \prod_{m=-\infty}^{\mathcal{L}_i(\beta)} (c_1(\mathcal{L}_i) + m\hbar) \prod_\rho \prod_{m=-\infty}^{0} (D_\rho + m\hbar)}{\prod_{i=1}^{\ell} \prod_{m=-\infty}^{0} (c_1(\mathcal{L}_i) + m\hbar) \prod_\rho \prod_{m=-\infty}^{D_\rho(\beta)} (D_\rho + m\hbar)}. \tag{11.73}$$

where $q_i = e^{t_i}$ and $q^\beta = \Pi_{i=1}^r q_i^{\int_\beta T_i}$. Note that if Σ is the standard fan for $\mathbb{P}^n$, then we recover (11.38). Turning to $J_\mathcal{V}$, we define

$$J_\mathcal{V} = e^{(t_0+\delta)/\hbar}\,\mathrm{Euler}(\mathcal{V}) \times$$
$$\Big(1 + \sum_{\beta \neq 0} q^\beta PD^{-1} e_{1*}\Big(\frac{\mathrm{Euler}(\mathcal{V}'_{\beta,2,1})}{\hbar - c_1(\mathcal{L}_1)} \cap [\overline{M}_{0,2}(X_\Sigma, \beta)]^{\mathrm{virt}}\Big)\Big), \tag{11.74}$$

where $[\overline{M}_{0,2}(X_\Sigma, \beta)]^{\mathrm{virt}}$ is the virtual fundamental class of $\overline{M}_{0,2}(X_\Sigma, \beta)$ and PD is Poincaré duality. Note that when X_Σ is convex, $[\overline{M}_{0,2}(X_\Sigma, \beta)]^{\mathrm{virt}}$ is just the usual fundamental class and the formula for $J_\mathcal{V}$ can be simplified. For example, when X_Σ is the convex variety $\mathbb{P}^n$, (11.74) reduces to (11.52).

In this situation, the variables q_i have degrees. As in Section 11.2.2, we define $\deg q_i$ by the equation

$$c_1(X_\Sigma) - c_1(\mathcal{V}) = \sum_{i=1}^{r} (\deg q_i)\, T_i.$$

We will assume that $X \subset X_\Sigma$ is a nef complete intersection in the sense of Section 5.5.3, which means that $-(K_{X_\Sigma} + \sum_{i=1}^{\ell} \mathcal{L}_i)$ is nef on X_Σ. When this occurs, we will assume that the basis $T_1, \dots, T_r$ of $H^2(X_\Sigma, \mathbb{Z})$ has been chosen so that

$-(K_{X_\Sigma} + \sum_{i=1}^{\ell} \mathcal{L}_i)$ lies in the cone generated by the T_i. This can always be arranged in the nef case. It follows that $\deg q_i \geq 0$ for all i.

We can now state Givental's version of the Toric Mirror Theorem.

THEOREM 11.2.16. *Let $X \subset X_\Sigma$ be a nef complete intersection, and let $I_\mathcal{V}$ and $J_\mathcal{V}$ be as in* (11.73) *and* (11.74). *Then $I_\mathcal{V}$ and $J_\mathcal{V}$ coincide after a triangular weighted homogeneous change of variables:*

$$t_0 \mapsto t_0 + f_0(q)\hbar + h(q), \quad t_i \mapsto t_i + f_i(q) \ \ \textit{for } 1 \leq i \leq r,$$

where $f_0, f_1 \dots, f_k, h$ are weighted homogeneous power series and $\deg f_0 = \deg f_i = 0,\ \deg h = 1$.

Ref: [Cox, Katz] Mirror Symmetry and algebraic geometry

Kähler - Einstein metric. Δ = fan Futaki invariants

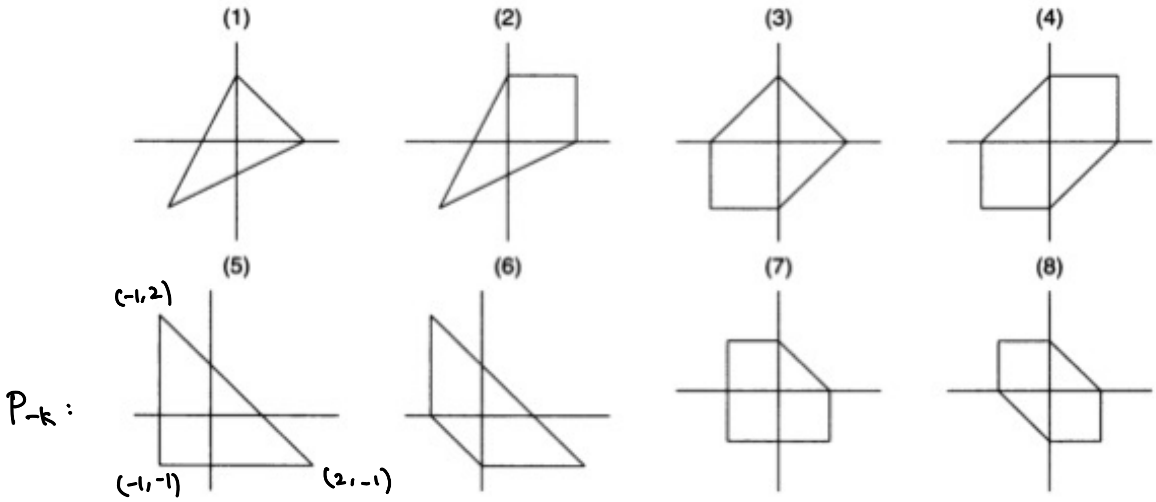
Theorem 5.1. *Let f_ω be the real-valued C^∞ function on Y defined uniquely, up to constant, by $Ric(\omega)-\omega=\sqrt{-1}\,\partial\bar\partial f_\omega$. Put $c:=((2\pi\, c_1(Y))^n[Y])^{-1}$, where $n=\dim_{\boldsymbol{C}} Y$. We further define a linear map $F=F_Y\colon \mathfrak{X}(Y)\to\boldsymbol{R}$ by*

$$F(V):=c\,\mathrm{Re}\left(\int_Y (Vf_\omega)\,\omega^n\right),\quad V\in\mathfrak{X}(Y).$$

Then this map F does not depend on the choice of ω. Moreover,
(a) *F is trivial on the commutator subalgebra of $\mathfrak{X}(Y)$.*
(b) *If Y admits an Einstein-Kähler form, then F is trivial.*

Corollary 5.5. *Let G be a nonsingular toric Fano variety such that* $\mathrm{Aut}(G_\Delta)$ *is reductive. Then $F\colon \mathfrak{X}(G_\Delta)\to\boldsymbol{R}$ is trivial if and only if $\boldsymbol{a}_\Delta=0$.*

a_Δ = barycenter of P_{-K}.
$-K = D_1+\cdots+D_r$



Ref: [Toshiki Mabuchi]

Einstein - Kähler forms, Futaki invariants and convex geometry on toric Fano varieties.